ME550 Notes

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December 22, 2021

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Chapter 1

Mathematical Preliminaries

1.1 Vector Spaces, Basis Sets, Summation Convention

1.1.1 Axiomatic Approach

We work with quantities like vectors in the 3D world. We would like to work on the 3D Euclidean vector space $\mathbb{R}^3$. Magnitude and distance between objects comes from the Euclidean properties. Real analysis stars with a concept of a field. A field is a bunch of quantities that satisfies numbers of axioms. These numbers are the foundation of fields. 14 axioms define the real numbers $\mathbb{R}$. From the definition of real numbers we want to define vectors. Vectors are more general tho, they can be functions. The definition of distance and a magnitude in a vector space can differ tho. These definitions comes from inner product operator. For example dot product is a measure of how two vectors are conforming each other. A norm of the vector space is the definition of a length of a quantity in the Vector space. To measure the distance we can define a metric over the vector spaces, this does not have to be independently defined, it can be induced from the norm of the vector space.

For the Euclidean inner (dot) product: $a \cdot b = \sum_i a_i b_i$

For the Euclidean norm: $||a|| = \sum_i a_i^2$

For the Euclidean metric: $||a - b||$

1.1.2 Vectors

Basis Sets

Consider $\sum_i^n \alpha_i v_i = 0$ where $v_i \neq 0$, then v_i are linearly independent iff $\alpha_i = 0 \forall i$, linearly dependent otherwise.

If a vector space V can accommodate at most n linearly independent vector then it has dimensions n: $\mathbb{R}^\ltimes$.

Any choice of n linearly independent vectors constitutes a basis for n dimensional vector space.

v_i are orthogonal if $v_i \cdot v_j = 0$ unless $i = j$

v_i is normalized if $||v_i|| = 1 \;\; \forall i$

Orthonormal basis set e_i are both orthogonal and normalized. Any vector in the vector space can be expressed like this:

$$a = \sum_i^n a_i e_i$$

where $a_i = a \cdot e_i$.

Kronecker Delta:$\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Which can be expressed like $\delta_{ij} = e_i \cdot e_j$

In this course we are by default work with orthonormal basis. We can go from not orthonormal basis to an orthonormal one with grahm-schmidt orthonormalization.

Summation Convention(Einstein Notation)

$$a \cdot b = \sum_i a_i b_i = a_i b_i$$

Rules:

- repeated "dummy" index indicates summation, others are free indices.
- no index may appear more than twice and indicate summation

$$A_{11}\alpha_1 + A_{12}\alpha_2 + A_{13}\alpha_3 = \sum_{i=1}^3 A_{1i}\alpha_i = A_{1i}\alpha_i$$

$$A_{11}B_{11} + A_{12}B_{21} + A_{13}B31 = A_{1i}B_{i1}$$

$$AB = C,\ C_{ij} = A_{ik}B_{kj}$$

Substitution Property of Kronecker Delta: $\delta_{ij}A_{jk} = A_{ik}$

$$\delta_{ij}A_{ij} = A_{ii}$$

1.2 Triple Product, Linear Operators, Tensor Product

Vector Product

A basis can be right or left handed.

Vector (cross) product:

$$e_i \times e_j = e_{ijk}e_k$$

Permutation symbol:

$$e_{ijk} = \begin{cases} 1 & \text{if ordered in clockwise + even permutations} \\ -1 & \text{if odd permutations} \\ 0 & \text{otherwise} \end{cases}$$

Note that $e_{ijk} = -e_{jik} = e_{jki}$

$$
\begin{aligned}
a \times b &= (a_i e_i) \times (b_j e_j) \\
&= a_i b_j e_i \times b_j \\
&= a_i b_j e_{ijk} e_k \\
&= a_i b_j e_{kij} e_k
\end{aligned}
$$

$$
\begin{aligned}
a \cdot (a \times b) &= (a_l e_l) \cdot (a_i e_i \times b_j e_j) \\
&= a_l a_i b_j e_{ijk} e_l \cdot e_k \\
&= a_i b_j a_k e_{ijk} \\
&= a_k b_j a_i e_{kji} \\
&= -a_i b_j a_k e_{ijk} = 0
\end{aligned}
$$

Triple Product

$$[a, b, c] = a \cdot (b \times c)$$

Triple product behaves like e_{abc} with respect to change of orders:

$$[a, b, c] = -[b, a, c] = [b, c, a]$$

$$[a, a, c] = 0$$

$$[a, b, c] = 0 \longmapsto \{a, b, c\} \text{ are linearly dependent}$$

1.2.1 Tensor Algebra

An operator $\underline{\mathrm{A}}$ uniquely maps $\underline{a} \in \mathbb{R}^n \to \underline{b} \in \mathbb{R}^n : \underline{Aa} = \underline{b}$
$\underline{\mathrm{A}}$ is linear if

1. $\underline{A}(\alpha \underline{a}) = \alpha(\underline{Aa})$
2. $\underline{A}(\underline{a} + \underline{b}) = \underline{Aa} + \underline{Ab}$
   Combine these two with how it acts on scalars and other linear operators.
3. Distributivity
4. Associativity
5. Identity tensor $\underline{\mathrm{I}}$
6. Zero tensor $\underline{0}$

We refer to this linear operator as a tensor.

$$
\begin{cases}
\text{in Particular, } \underline{\mathrm{A}} \text{ is a 2nd order tensor (matrix)} \\
\text{Specifically, } \underline{\mathrm{a}} \text{ is a 1st order tecnsor (vector)}
\end{cases}
$$

Examples for a second order tensors are stress, strains.

1.2.2 Tensor Product

Let $\underline{a} = a_i \underline{e_i} \to \{a\} = \{a_1, a_2, ...\}$ (array) and let $\underline{b}$ similar. Then

$$\{a\}^T\{b\} = a_i b_i = a \cdot b$$

.

$$\{a\}\{b\}^T = \begin{bmatrix} a_1 b_1 & \dots \\ \vdots & \end{bmatrix} = [A]$$

$$A_{ij} = a_i b_j$$

A particular type of 2nd order tensor: $\underline{a} \otimes \underline{b}$

$$\begin{aligned} \underline{a} \otimes \underline{b} &= (a_i \underline{e_i}) \otimes (b_j \underline{e_j}) \\ &= \underbrace{a_i b_j}_{\text{components}} \underbrace{\underline{e_i} \otimes \underline{e_j}}_{\text{basis for 2nd order tensor}} \end{aligned}$$

Operation on a vector:

$$(\underline{a} \otimes \underline{b})\underline{c} = (\underline{b} \cdot \underline{c})\underline{a}$$

In general

$$\underline{A} = A_{ij} \underline{e_i} \otimes \underline{e_j}$$

1.3 Tensors, Symmetry, Invariants

To find the corresponding entries of a tensor use following

$$A_{ij} = \underline{e_i} \cdot \underline{A} \underline{e_j}$$

4th order tensors will appear in elasticity but not that prevalent in the rest of it. For example

$$\underline{(}\mathbb{C}) = \underline{A} \otimes \underline{B} \to C_{ijkl} = A_{ij} B_{kl}$$

$$\underline{\mathbb{C}} = C_{ijkl} \underline{e_i} \otimes \underline{e_j} \otimes \underline{e_k} \otimes \underline{e_l}$$

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{Ia} = (\underline{e_i} \otimes \underline{e_i})(a_j \underline{e_j}) = a_j (\underline{e_i} \cdot \underline{e_j}) \cdot \underline{e_i} = \underline{a}$$

For the tensor operation on another tensor we did not define any constraint yet but we can show the consistency of the previous theorem with the following example:

$$
\begin{aligned}
\underline{AB} &= \underline{C}\\
(A_{ij}\underline{e_i}\otimes\underline{e_j})(B_{kl}\underline{e_k}\otimes\underline{e_l})\underline{c} &= \underline{C}\underline{c}\\
(A_{ij}\underline{e_i}\otimes\underline{e_j})(B_{kl}c_l\underline{e_k}) &= \underline{C}\underline{c}\\
(A_{ij}B_{jl}c_l\underline{e_i}) &= C_{il}c_l\underline{e_i} \rightarrow A_{ij}B_{jl} = C_{il}
\end{aligned}
$$

Alternatively we can realize a rule:

$$(\underline{a}\otimes\underline{b})(\underline{c}\otimes\underline{d}) = (\underline{b}\cdot\underline{c})\underline{a}\otimes\underline{d}$$

For the transpose effect on these results is as the following:

$$\underline{A}^T\underline{B} = \underline{C} \rightarrow C_{ij} = A_{ki}B_{kj}$$

$$\underline{AB}^T = \underline{C} \rightarrow C_{ij} = A_{ik}B_{jk}$$

1.3.1 Transpose/Symmetry

component/extrinsic representation:

$$A_{ij}^T = A_{ji} \text{ this comes with a caveat}$$

intrinsic representation

$$\underline{a}\cdot\underline{Ab} = \underline{A}^T\underline{a}\cdot\underline{b}$$

Symmetric tensor

$$\underline{A}^T = \underline{A}$$

$$(\underline{a}\otimes\underline{b})^T = \underline{b}\otimes\underline{a}$$

$$\underline{A}^T = A_{ji}\underline{e_i}\otimes\underline{e_j}$$

$$\underline{A}^T\underline{a} = \underline{b} \rightarrow b_i = A_{ji}a_j$$

Skew symmetric tensor:

$$\underline{A}^T = -\underline{A}$$

$$A_{ji} = -A_{ij}$$

A tensor has 9 defining components, a symmetric tensor has 6 and a skew symmetric tensor has 3 defining component(diagonals should be 0) and this can be used as a vector in a sense. Notice this is true for 2nd order tensor. For any tensor, once can define:

$$\underline{A} = \underline{A}^{sym} + \underline{A}^{skw} \begin{cases} \underline{A}^{sym} = \frac{1}{2}(\underline{A}+\underline{A}^T) \\ \underline{A}^{skw} = \frac{1}{2}(\underline{A}-\underline{A}^T) \end{cases}$$

1.3.2 Invariants

For any $\underline{A}$ operating on $\mathbb{R}^3$ we define 3 scalars: invariants. Choose any set of linearly independent vectors: $\{\underline{a}, \underline{b}, \underline{c}\}$

$$I_A = \frac{[\underline{Aa}, \underline{b}, \underline{c}] + [\underline{a}, \underline{Ab}, \underline{c}] + [\underline{a}, \underline{b}, \underline{Ac}]}{[\underline{a}, \underline{b}, \underline{c}]} = tr[A] = A_{ii}$$

$$II_A = \frac{[\underline{Aa}, \underline{Ab}, \underline{c}] + [\underline{Aa}, \underline{b}, \underline{Ac}] + [\underline{a}, \underline{Ab}, \underline{Ac}]}{[\underline{a}, \underline{b}, \underline{c}]} = \frac{1}{2}[A_{ii}^2 - A_{ij}A_{ji}] = \frac{1}{2}[tr[A]^2 - tr[A^2]]$$

$$III_A = \frac{[\underline{Aa}, \underline{Ab}, \underline{Ac}]}{[\underline{a}, \underline{b}, \underline{c}]} = det[A] = e_{ijk}A_{i1}A_{j2}A_{k3}$$

1.4 Inverse, Eigenvalue Problem

1.4.1 Inner/Dot/Scalar product on Tensors

$$\underline{A} \cdot \underline{B} = tr[\underline{AB}^T] \text{ (intrinsic)}$$

Since $\underline{AB}^T = \underline{C}$ with $C_{ij} = A_{ik}B_{jk}$ and $tr[\underline{C}] = C_{ii}A_{ik}B_{ik}$.

A spherical tensor $\underline{A}$ is such that $\underline{A} = \alpha\underline{I}$

A devitoric tensor $\underline{A}$ is such that $tr[\underline{A}] = 0$

The spherical part of a tensor is $\underline{A}^{sph} = \frac{1}{3}tr[A]\underline{I}$. This 1/3 assumes we are in 3 dimension for a 2nd order tensor.

The deviatoric part of a tensor is $\underline{A}^{dev} = \underline{A} - \underline{A}^{sph}$

1.4.2 Inverse and Cofactor

$\underline{A}^{-1}$ satisfies $\underline{A}^{-1}\underline{A} = \underline{AA}^{-1} = \underline{I}$.

$$A_{ik}^{-1}A_{kj} = A_{ik}A_{kj}^{-1} = \delta_{ij}$$

Inverse exists iff $det[\underline{A}] \neq 0$.

Cofactor of a tensor $\underline{A}^{\#}$

$$\underline{Aa} \times \underline{Ab} = \underline{A}^{\#}(\underline{a} \times \underline{b})$$

When inverse exists $\underline{A}^{\#} = det[\underline{A}]\underline{A}^{-T} \rightarrow \underline{A}^{-1} = \frac{1}{det[\underline{A}]}\overbrace{\underline{A}^{\#,T}}^{\text{adjugate}}$

$[\underline{A}^{\#}]$: cofactor matrix

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \underline{A}^{\#} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 9 \\ 0 & 2 & 5 \end{bmatrix} \rightarrow \underline{A}^{\#} = \begin{bmatrix} -8 & -15 & 6 \\ 8 & 5 & -2 \\ -8 & 3 & 2 \end{bmatrix}$$

Notice that $II_A = tr[\underline{A}^{\#}]$ in the case of last operator the invariants are:

$$\begin{aligned} I_A &= 8 \\ II_A &= -1 \\ III_A &= 16 \end{aligned}$$

1.4.3 Eigenvalue Problem

We seek vector $\underline{v}$ for $\underline{A}$ such that

$$\underline{A}\underline{v} = \lambda\underline{v}$$

For non-trivial solutions

$$(\underline{A} - \lambda\underline{I})\underline{v} = 0 \rightarrow det(\underline{A} - \lambda\underline{I}) = 0$$

For 3D and 2nd order tensors the solution comes from the char equation:

$$\lambda^3 - I_A\lambda^2 + II_A\lambda - III_A = 0$$

Cayley-Hamilton Theorem

A tensor satisfies its own characteristic equation:

$$\underline{A}^3 - I_A\underline{A}^2 + II_A\underline{A} - III_A\underline{I} = \underline{0}$$

which defines the inverse from another way:

$$\underline{A}^{-1} = \underbrace{\frac{1}{III_A}}_{1/det[A]} \underbrace{[\underline{A}^2 - I_A\underline{A} + II_A\underline{I}]}_{\underline{A}^{\#,T}}$$

$$\begin{aligned} 0 &= det[\underbrace{\underline{A} - \lambda\underline{I}}_{\underline{B}}] \\ &= \frac{[\underline{B}\underline{a}, \underline{B}\underline{b}, \underline{B}\underline{c}]}{[\underline{a}, \underline{b}, \underline{c}]} \\ &= \frac{[\underline{A}\underline{a}, \underline{A}\underline{b}, \underline{A}\underline{c}] + [\underline{A}\underline{a}, -\lambda\underline{b}, \underline{A}\underline{c}] + [-\lambda\underline{a}, \underline{A}\underline{b}, \underline{A}\underline{c}] + [\underline{A}\underline{a}, \underline{A}\underline{b}, -\lambda\underline{c}] + [\underline{A}\underline{a}, -\lambda\underline{b}, -\lambda\underline{c}] + \dots}{[\underline{a}, \underline{b}, \underline{c}]} \\ &\frac{+[-\lambda\underline{a}, \underline{A}\underline{b}, -\lambda\underline{c}] + [-\lambda\underline{a}, -\lambda\underline{b}, \underline{c}] + [-\lambda\underline{a}, -\lambda\underline{b}, -\lambda\underline{c}]}{[\underline{a}, \underline{b}, \underline{c}]} \\ &= III_A - \lambda II_A + \lambda^2 I_A - \lambda^3 \end{aligned}$$

$$\text{If } \underline{A} \text{ is symmetric then } \begin{cases} \lambda_i \in \mathbb{R} \\ \underline{v_i} \in \mathbb{R}^3 \\ \lambda_i \neq \lambda_j \rightarrow \underline{v_i} \cdot \underline{v_j} = 0 \end{cases}$$

$\{\lambda_1, \lambda_2, \lambda_3\}$: spectrum of $\underline{A}$. Spectral decomposition of this tensor is:

$$\underline{A} = \sum_{i=1}^{3} \lambda_i \underline{v_i} \otimes \underline{v_i}$$

in other words $\underline{A} = \underline{AI} = \underline{A}(\underline{v_i} \otimes \underline{v_i}) = (\underline{A}\underline{v_i}) \otimes \underline{v_i} = \sum_i \lambda_i(\underline{v_i} \otimes \underline{v_i})$

A symmetric tensor $\underline{A}$ is positive $\begin{cases} definite \\ semi-definite \end{cases}$ if $\begin{cases} \lambda_i > 0 \\ \lambda_i \geq 0 \end{cases}$ negative is sign reversed. Alternatively $\underline{A}$ sym is pos def if $\underline{a} \cdot \underline{Aa} > 0 \; \forall \underline{a} \neq 0$

if $\underline{A}$:pos def $\underline{A}^{1/2} = \sum_i \sqrt{\lambda_i} \underline{v_i} \otimes \underline{v_i}$

if $\underline{A}$:non semi-def $\underline{A}^{-1} = \sum_i \lambda_i^{-1} \underline{v_i} \otimes \underline{v_i}$

We can also express invariants with eigenvalues:

$$I_A = \lambda_1 + \lambda_2 + \lambda_3$$
$$II_A = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$
$$III_A = \lambda_1\lambda_2\lambda_3$$

Axial Vector

$\underline{W}$:skew $\rightarrow \underline{W}^T = -\underline{W}$:3 independent components like a vector. Let $\underline{Wv} = \underline{v}' \rightarrow \underline{w} \times \underline{v} = \underline{v}'$. Here $\underline{w}$ is the axial vector of $\underline{W}$: $\underline{Wv} = \underline{w} \times \underline{v}$

$det[\underline{W}] = det[-\underline{W}^T] = (-1)^3 det[\underline{W}^T] = -det[\underline{W}] \rightarrow det[\underline{W}] = 0$ which means that at least 1 eigenvalue is zero.

1.5 Polar Decomposition, Change of Basis, Grad-Div-Curl

Orthogonal Tensor

An orthogonal tensor is not quite skew, not symmetric as well. $\underline{Q}^T = \underline{Q}^{-1} \rightarrow \underline{QQ}^T = \underline{I}$

$$\det[\underline{Q}] = \begin{cases} 1 & \text{proper orthogonal tensor} \\ -1 & \text{improper orthogonal tensor} \end{cases}$$

These tensors has at least one eigenvalue as 1 $\underline{Q}\underline{a} = \underline{a}$. For the proper orthogonal tensor you can obtain rotations but with the improper orthogonal tensor you get a mirror like behaviour.

1.5.1 Polar Decomposition

Take any invertible tensor $\underline{E}$ admits two particular decomposition:

$\underline{E} = \underline{VR}$:left polar decomposition

$\underline{E} = \underline{RU}$:right polar decomposition

Here $\underline{U}$, $\underline{V}$ are symmetric positive definite tensors and $\underline{R}$ is orthogonal tensor.

1.5.2 Change of Basis

2 sets of orthonormal basis $\underline{e_i}$, $\underline{e_i'} = \underline{Q}\underline{e_i} \rightarrow$ if $\underline{Q}$ is given.

$$\underline{Q} = Q_{mn}\underline{e_m} \otimes \underline{e_n} \rightarrow Q_{mn} = \underline{e_m} \cdot \underline{Q}\underline{e_n} = \underline{e_m} \cdot \underline{e_n'}$$
$$\underline{Q} = \underline{Q}\underline{I} = \underline{Q}\underline{e_i} \otimes \underline{e_j} = \underline{(Qe_i)} \otimes \underline{e_j} = \underline{e_i'} \otimes \underline{e_j}$$
$$\text{so } \underline{Q} = \begin{bmatrix} Q_{11} & Q12 & \dots \\ \vdots & \ddots & \end{bmatrix} \text{ in the } \underline{e_i} \otimes \underline{e_j}$$
$$\text{but } \underline{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ in the } \underline{e_i'} \otimes \underline{e_j}$$

Let $\underline{a} = a_i'\underline{e_i'} = a_m\underline{e_m} = a_i'\underline{Q}\underline{e_i} = a_i'(Q_{mn}\underline{e_m} \otimes \underline{e_n})\underline{e_i} = a_iQ_{mi}\underline{e_m} \rightarrow a_m = Q_{mi}a_i'$ or $a_i' = Q_{im}a_m$

$$\underline{e_i} \overset{Q}{\rightarrow} \underline{e_i} \quad \{a\} \overset{Q^T}{\longrightarrow} \{a'\}$$

Compare with a physical rotation $\underline{b} = \underline{Q}\underline{a}$:

$$\begin{aligned} \underline{b} &= b_i'\underline{e_i'} \\ &= \underline{Q}(a_i\underline{e_i}) \\ &= a_i\underline{e_i'} \rightarrow b_i' = a_i \end{aligned}$$

But

$$\begin{aligned} \underline{b} &= b_m\underline{e_m} \\ &= (Q_{mn}\underline{e_m} \otimes \underline{e_n})(a_i\underline{e_i}) \\ &= Q_{mi}a_i\underline{e_m} \rightarrow b_m = Q_{mi}a_i \end{aligned}$$

Hence

$$\underline{e_i} \overset{Q}{\rightarrow} \underline{e_i} \quad \{a\} \overset{Q}{\rightarrow} \{b\}$$

It is important to distinguish between the change in interpretation vs the change in perceived physics.

For tensors one can derive similar results:

(i) Change of basis, $\underline{A} = A_{ij}(\underline{e_i} \otimes \underline{e_j}) = A_{ij}' = \underline{e_i} \otimes \underline{e_j}$

$$\underline{A'} = \underline{Q}^T\underline{A}\underline{Q} \text{ or } A_{ij}' = Q_{ik}^T A_{kl} Q_{lj} = Q_{ki} A_{kl} Q_{lj}$$

(ii) Rotation,

$$\underline{B} = \underline{Q}\underline{A}\underline{Q}^T \text{ or } B_{ij} = Q_{im}A_{mn}\underline{Q_{nj}}^T = Q_{im}A_{mn}Q_{jn}$$

Let $\underline{Aa} = \underline{b}$

$$\begin{aligned}
&\underline{Q}\underline{A}(\underline{Q}^T\underline{Q})\underline{a} = \underline{Qb} \\
&(\underline{Q}\underline{A}\underline{Q}^T)\underline{Qa} = \underline{Qb} \\
&(\underline{B})\underline{Qa} = \underline{Qb}
\end{aligned}$$

Here $\underline{\text{B}}$ maps the rotation of $\underline{a}$ to rotation of $\underline{b}$.

1.5.3 Tensor Calculus

We will have a domain, in general this will be a 3D object. We will indicate a point P on or in the object with $\underline{x}$. Domain will be represented with $\mathbb{D}$ and boundary will be $\partial\mathbb{D}$(surface) and $\underline{n}$ as the outward unit normal. We will have scalar/vectors/tensors as functions $\phi(\underline{x}, t)$, $\underline{v}(\underline{x}, t)$, $\underline{T}(\underline{x}, t)$.

Gradient

One tensor level up

$$\begin{aligned}
\nabla\phi &= \text{grad}[\phi] = \frac{\partial\phi}{\partial\underline{x}} = \frac{\partial\phi_i}{\partial x_i}\underline{e_i} = \phi_{,i}\,\underline{e_i} \\
\nabla\underline{v} &= \text{grad}[\underline{v}] = \frac{\partial\underline{v}}{\partial\underline{x}} = \frac{\partial v_i}{\partial x_j}\underline{e_i} \otimes \underline{e_j} = v_{i,j}\underline{e_i} \otimes \underline{e_j} \\
\nabla\underline{T} &= \text{grad}[\underline{T}] = \frac{\partial\underline{T}}{\partial\underline{x}} = \frac{\partial T_{ij}}{\partial x_k}\underline{e_i} \otimes \underline{e_j} \otimes \underline{e_k} = T_{i,j,k}\underline{e_i} \otimes \underline{e_j} \otimes \underline{e_k}
\end{aligned}$$

Divergence

One tensor level down

$$\text{div}[\underline{v}] = \text{tr}[\text{grad}[\underline{v}]] = \frac{\partial v_i}{\partial x_i} = v_{i,i} \rightarrow \nabla = \frac{\partial}{\partial x_i}\underline{e_i}, \quad \text{div}[\underline{v}] = \nabla \cdot \underline{v}$$

$$\underbrace{\text{div}[\underline{T}]}_{\underline{t}} \cdot \underline{a} = \text{div}[\ \underbrace{\underline{T}^T\underline{a}}_{T_{ki}a_k\underline{e_i}}\ ]\ \forall\underline{a}:\ \text{constant}$$

$$\begin{aligned}
(t_k\underline{e_k}) \cdot \underline{a} &= (T_{ki}a_k)_{,i} \\
t_k a_k &= T_{ki,i}a_k \rightarrow t_k = T_{ki,i} \\
\text{div}[\underline{T}] &= T_{ki,i}\underline{e_k} \\
&= T_{ij,j}\underline{e_i} \\
&= \frac{\partial T_{ij}}{\partial x_j}\underline{e_i}
\end{aligned}$$

Curl

Stays in the same tensor level.

$$\underbrace{\text{curl}[\underline{v}]}_{\underline{c}} \cdot \underline{a} = \text{div}[\underline{v} \times \underline{a}] \;\; \forall \underline{a} : \;\text{constant}$$

$$\text{curl}[\underline{T}] \cdot \underline{a} = \text{curl}[\underline{T}^T \underline{a}]$$

$$\begin{aligned}
c_j \underline{e_j} \cdot \underline{a} &= \text{div}[\underbrace{e_{ijk} v_i a_j}_{d_k} \underline{e_k}] \\
c_j a_j &= d_{k,k} \\
c_j a_j &= e_{ijk} v_{i,k} a_j \to c_j = e_{ijk} v_{i,k} \\
\text{curl}[\underline{v}] &= c_j \underline{e_j} \\
&= e_{ijk} v_{i,k} \underline{e_j} \\
&= e_{jki} v_{i,k} \underline{e_j} \\
&= \nabla \times \underline{v}
\end{aligned}$$

1.6 Integral Theorems

1.6.1 Integral Theorem

Derivative operation can be generalized $\phi(\underline{v}) \to \frac{\partial \phi}{\partial v_i} \underline{e_i}$ or $\phi(\underline{T}) \to \frac{\partial \phi}{\partial T_{ij}} \underline{e_i} \otimes \underline{e_j}$. In general $\underline{A}(\underline{B}) \to \frac{\partial A_{ij}}{\partial B_{kl}} \underline{e_i} \otimes \underline{e_j} \otimes \underline{e_k} \otimes \underline{e_l}$. From these we can derive identities: $\frac{\partial \underline{v}}{\partial \underline{v}} = \underline{I}$, $\frac{\partial \underline{A}}{\partial \underline{A}} = \underline{II}$(fourth order identity tensor)

In the integrals we generally want to convert volume integration to the surface integration. These 4 theorems fit into a stencil and this stencil is as follows:

$$\int_{\mathbb{D}} (\ldots)_{,k} dv = \int_{\partial \mathbb{D}} (\ldots) n_k da$$

Assume continuous functions $\phi / \underline{v} / \underline{T}$ Volume boundary is a surface and surface boundary is a line.

(1)

$$\int_{\mathbb{D}} \text{grad}[\phi] dv = \int_{\partial \mathbb{D}} \phi \underline{n} da$$

$$\int_{\mathbb{D}} \phi_{,i} dv = \int_{\partial \mathbb{D}} \phi n_i da$$

(2)

$$\int_{\mathbb{D}} \mathrm{grad}[\underline{v}]dv = \int_{\partial\mathbb{D}} \underline{v} \otimes \underline{n} da$$

$$\int_{\mathbb{D}} v_{i,j} dv = \int_{\partial\mathbb{D}} v_i n_j da$$

(3) Gauss-Ostragradsky Theorem(divergence theorem)

$$\int_{\mathbb{D}} \mathrm{div}[\underline{v}]dv = \int_{\partial\mathbb{D}} \underline{v} \cdot \underline{n} da$$

$$\int_{\mathbb{D}} v_{i,i} dv = \int_{\partial\mathbb{D}} v_i n_i da$$

(4)

$$\int_{\mathbb{D}} \mathrm{div}[\underline{T}]dv = \int_{\partial\mathbb{D}} \underline{Tn} da$$

$$\int_{\mathbb{D}} T_{ij,j} dv = \int_{\partial\mathbb{D}} T_{ij} n_j da$$

Chapter 2

Kinematics

2.1 Configuration and Motion

Kinematics is analysis of motion and deformation. Admit an observer(Ω) which is the frame of reference. Let there be a body $\mathbb{B}$, reference(Initial(t=0), undeformed(stress-free)) configuration $\mathbb{R}_0$ and current/spatial(deformed) configuration $\mathbb{R}$. M is on the body, $\mathbb{P}$ is on the $\mathbb{R}$ and $\mathbb{P}_0$ is on the $\mathbb{R}_0$:

$$\chi_0(M) : \mathbb{P}_0 \leftrightarrow M$$
$$\chi_t(M) : \mathbb{P} \leftrightarrow M$$
$$\chi_t(\underline{X}) : \mathbb{P}_0 \leftrightarrow \mathbb{P}$$

A body is a collection of particles. This body will move and deform with time. Every particle in this body or say list we know its position($\underline{x}$) with the $\chi_t(M)$. Identify each M with $\underline{X} = \chi_0 M$ and $\underline{x} = \chi_t(M)$ position vectors. ξ : Euclidean Point Space, $\mathbb{P}_0, \mathbb{P} \in \xi$ and motion/deformation map is $\underline{x} = \chi_t(\underline{X})$(path of a particle).

$$\begin{cases} \text{velocity:} & \underline{v} = \dfrac{\partial \underline{\chi}_t(M)}{\partial t} = \underline{v}(M,t) \\ \text{acceleration:} & \underline{a} = \dfrac{\partial \underline{v}(M,t)}{\partial t} = \underline{a}(M,t) \end{cases}$$

For clarity/precision distinguish configuration basis sets:

$$\begin{cases} \underline{X} = X_A \underline{E}_A \\ \underline{x} = x_i \underline{e_i} \end{cases}$$

Similarly for operators:

$$\begin{cases} \text{Grad}[\underline{v}] = \dfrac{\partial \underline{v}}{\partial \underline{X}} = \dfrac{\partial v_i}{\partial X_A} \underline{e_i} \otimes \underline{E_A} \\ \text{grad}[\underline{v}] = \dfrac{\partial \underline{v}}{\partial \underline{x}} = \dfrac{\partial v_i}{\partial x_j} \underline{e_i} \otimes \underline{e_j} \end{cases}$$

Similar with the divergence and curl: Div[$\underline{v}$], Curl[$\underline{v}$].

2.2 Lagrangian and Eulerian Representations, Material Time Derivative

$\chi_t(\underline{X})$ is the motion map of the particles. Here this will be satisfied:

$$\underline{v} = \frac{\partial \chi_t(M)}{\partial t} = \frac{\partial \chi_t(\underline{X})}{\partial t}$$

2.2.1 Lagrangian and Eulerian Representations

$$f = \begin{cases} \hat{f}(x) = 2x^2 \\ \check{f}(y) = 2y \end{cases} \rightarrow \hat{f}(2) = \check{f}(4) = 8$$

In general a function may have different representations

$$\phi = \check{\phi}(M,t) = \hat{\phi}(\underline{X},t) = \tilde{\phi}(\underline{x},t)$$

$\check{\phi}$: we record the value of ϕ for a given particle M at a given time. This is the material representation.

$\hat{\phi}$: we record the value of ϕ for a given particle M and that is associated(labeled) with $\underline{X}$. This is called the lagrangian representation.

$\tilde{\phi}$: we record the value of ϕ for a given particle M that happens to occupy the position specified by the $\underline{x}$ at time t. This is the Eulerian(Spatial) representation.

Material Time Derivative

This is

$$\frac{d\phi}{dt} = \dot{\phi} = \begin{cases} \frac{\partial \check{\phi}}{\partial t} = \frac{\partial \phi(\check{M},t)}{\partial t}\Big|_{\text{M is fixed}} \\ \frac{\partial \hat{\phi}}{\partial t} = \frac{\partial \phi(\hat{\underline{X}},t)}{\partial t}\Big|_{\underline{X}\text{ is fixed}} \\ \frac{\partial \tilde{\phi}(\underline{x},t)}{\partial t} + \frac{\partial \tilde{\phi}(\underline{x},t)}{\partial x}\frac{dx}{dt} = \frac{\partial \tilde{\phi}}{\partial t} + \text{grad}[\phi] \cdot \underline{v} \end{cases}$$

Here $\frac{\partial \tilde{\phi}}{\partial t}$ is the rate of change of ϕ at a fixed point $\mathbb{P}$ in ξ at time t. This is also called the spatial/local time derivative.

$\text{grad}[\phi] \cdot \underline{v}$ is the rate of change of ϕ due to the motion of the particle which happens to occupy $\mathbb{P}$ at time t.

In particular:

$$\underline{a} = \dot{\underline{v}} = \frac{\partial \check{\underline{v}}(M,t)}{\partial t} = \frac{\partial \hat{\underline{v}}(\underline{X},t)}{\partial t} = \frac{\partial \tilde{\underline{v}}(\underline{x},t)}{\partial t} + \underbrace{\frac{\partial \tilde{\underline{v}}}{\partial \underline{x}}}_{\underline{L}\text{: velocity gradient tensor}} \cdot \underline{v}$$

$$\underline{L} = \underbrace{\underline{L}^{sym}}_{\underline{D}} + \underbrace{\underline{L}^{skw}}_{\underline{W}}$$

$\underline{D}$: Stretching or rate-of-spin tensor
$\underline{W}$: Spin or vorticity tensor

Temperature Sensors and a Fly

Lets say that fly(depicting a particle) has a $\underline{v}$ speed and it has a sensor attached to it and measuring the temperature around a room. We are interested in the how fast temperature changes as the fly moves around. In other words we want a material time derivative of temperature with respect to the fly. Since sensor is moving with the fly this represents the lagrangian representation. Let say that the velocity of the temperature is $\underline{v}_L$. The temperature signal coming from this sensor is $\hat{\phi}(\underline{X}, t)$.

Lets imagine also a set of sensors scattered around the room, their positions are fixed and this represents the eulerian representation. Signal from the this sensor will be $\tilde{\phi}\underline{x}, t$. We want $\dot{\phi}$ which is the temperature change that fly experiences:

$$\begin{aligned}\dot{\phi} &= \frac{d\phi}{dt} \\ &= \frac{\partial\hat{\phi}}{\partial t} \\ &= \frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial\tilde{\phi}}{\partial\underline{x}} \cdot \underline{v}\end{aligned}$$

Lets have a set of sensors that are moving with a velocity $\underline{v}_{ALE} \neq 0 \neq \underline{v}$. We want to have the information from these sensors $\bar{\phi}$ than we need to take this velocity into account:

$$\dot{\phi} = \frac{\partial\bar{\phi}}{\partial t} + \frac{\partial\bar{\phi}}{\partial\underline{x}} \cdot (\underline{v} - \underline{v}_{ALE})$$

This representation is neither lagrangian nor eulerian. This is called Arbitrary Lagrangian Eulerian(ALE) representation.

Particular types of motion

(1) Rigid body motion:

$$\underline{x} = \chi(\underline{X}, t) = \underbrace{\underline{Q}(t)\underline{X}}_{\text{rotation}} + \underbrace{\underline{c}(t)}_{\text{translation}} \qquad \underline{Q}: \text{ orthogonal}$$

$$\underline{v} = \dot{\underline{Q}}\underline{X} + \dot{\underline{c}}$$

(2) Steady motion/flow: At a point in space the attribute wont change. In general flow is not steady hence the spatial description has also time dependence.

$$\underline{v} = \hat{\underline{v}}(\underline{X}, t) = \tilde{\underline{v}}(\underline{x})$$

$$\underline{a} = \frac{\partial\tilde{\underline{v}}}{\partial t} + \frac{\partial\tilde{\underline{v}}}{\partial\underline{x}} \cdot \underline{v} = \frac{\partial\tilde{\underline{v}}}{\partial\underline{x}} \cdot \underline{v}$$

Lets assume that we know the motion map $\chi_t(\underline{X})$. Line elements can stretch and rotate, surface elements can stretch rotate, normal can change, volume elements can change their shape and their volumes as well. When deformation introduced we are going to touch on the strain element of materials as well.

2.3 Infinitesimal(Differential) Material Line/Surface/Volume

$\chi_t(\underline{X}) : \mathbb{R}_0 \to \mathbb{R}$ and $\underline{X} \in \mathbb{R}_0$, $\underline{x} \in \mathbb{R}$.

Now, lets have a line in $\mathbb{R}_0$ with length L going through coordinate S. Here S/s is the arclength parametrization on $\mathbb{R}_0/\mathbb{R}$. Then lets have a tangent to this line at the point $\underline{X}$. $d\underline{X}$. Lets have infinitesimal increment on the archlenghts: $dS = ||d\underline{X}||$ and $ds = ||d\underline{x}||$.

For a surface element we can take a patch with a small area dA with an outward normal vector $\underline{N}$, these elements will deform to be da and $\underline{n}$.

For a volume element we will have dV and dv.

Now, how do we go from $\mathbb{R}_0$ to $\mathbb{R}$.

2.3.1 Material Line Element

$d\underline{X} \to d\underline{x}$.

$$d\underline{X} = dS\underline{M} \text{ or } \frac{\partial \underline{X}}{\partial S} = \underline{M}$$

$$d\underline{x} = ds\underline{m} \text{ or } \frac{\partial \underline{x}}{\partial s} = \underline{m}$$

Now, $\underline{m} = \dfrac{\partial \underline{x}}{\partial s} = \dfrac{\partial \underline{x}}{\partial \underline{X}} \dfrac{\partial \underline{X}}{\partial S} \dfrac{\partial S}{\partial s} = \underline{F}\underline{M}\dfrac{dS}{ds}$ or $d\underline{x} = ds\underline{m} = \underline{F}(\underbrace{ds\underline{M}}_{d\underline{X}}) \to d\underline{x} = \underline{F}d\underline{X}$. Here $\underline{F}$ is the tensor that maps from $\underline{X} \to \underline{x}$. This tensor $\underline{F}$ is called the Deformation Gradient Tensor:

$$\underline{F} = \frac{\partial \underline{x}}{\partial \underline{X}} = \frac{\partial x_i}{\partial X_A} \underline{e_i} \otimes \underline{E_A} = F_{iA} \underline{e_i} \otimes \underline{E_A} = \hat{\underline{F}}(\underline{X}, t) = \tilde{\underline{F}}(\underline{x}, t)$$

Here $\underline{e_i} \otimes \underline{E_A}$ is a two-point tensors: one leg in $\mathbb{R}_0$ and other in $\mathbb{R}$.

2.3.2 Material Surface Element

Lets have a two vector element that defines a parallelogram: $d\underline{X_1}$, $d\underline{X_2}$. $\underline{N} = \dfrac{\underline{dX_1} \times \underline{dX_2}}{||\underline{dX_1} \times \underline{dX_2}||}$ and $dA = ||\underline{dX_1} \times \underline{dX_2}||$ and $d\underline{A} = dA\underline{N}$. Same configuration for the components in the $\mathbb{R}$. Since we know the relation for the line elements $d\underline{X_1}$ and $d\underline{X_2}$ we can figure out the relation for surface elements:

$$\begin{aligned} d\underline{a} &= da\underline{n} \\ &= \underline{dx_1} \times \underline{dx_2} \\ &= \underline{FdX_1} \times \underline{FdX_2} \\ &= \underline{F}^{\#}\underline{dX_1} \times \underline{dX_2} \end{aligned}$$

Here emerges the Nanson's formula:

$$\boxed{da\underline{n} = \underline{F}^{\#} dA\underline{N}}$$

If we dot both sides with itself:

$$da^2 = \underline{F}^{\#}\underline{N} \cdot \underline{F}^{\#}\underline{N} dA^2 = \underline{N}\underline{F}^{\#T} \cdot \underline{F}^{\#}\underline{N} dA^2 \rightarrow da = \underbrace{\sqrt{\underline{N}\underline{F}^{\#T} \cdot \underline{F}^{\#}\underline{N}}}_{\text{arela stretch}} dA$$

2.3.3 Material Volume Element

To construct a volume we need 3 line elements: $\underline{dX_1}$, $\underline{dX_2}$, $\underline{dX_3}$ for dV. Same configuration for the $\mathbb{R}$. Volume of this construct is

$$\begin{aligned} dV &= \underline{dX_3} \cdot (\underline{dX_1} \times \underline{dX_2}) \\ &= [d\underline{X_3}, d\underline{X_1}, d\underline{X_2}] \\ &= [d\underline{X_1}, d\underline{X_2}, d\underline{X_3}] \end{aligned}$$

This is the same for $\mathbb{R}$: $dv = [d\underline{x_1}, d\underline{x_2}, d\underline{x_3}]$

$$\begin{aligned} \frac{dv}{dV} &= \frac{[d\underline{x_1}, d\underline{x_2}, d\underline{x_3}]}{[d\underline{X_1}, d\underline{X_2}, d\underline{X_3}]} \\ &= \frac{[\underline{F}d\underline{X_1}, \underline{F}d\underline{X_2}, \underline{F}d\underline{X_3}]}{[d\underline{X_1}, d\underline{X_2}, d\underline{X_3}]} \\ &= \det[\underline{F}] = J \end{aligned}$$

So the next expression is the Jacobian of the Deformation/Mapping

$$J = \det[\underline{F}] = \frac{dv}{dV} = \hat{J}(\underline{X}, t) = \tilde{J}(\underline{x}, t) > 0$$

For physical deformations J should be greater than 0 because if it is less than 0 at some point it should go through 0 as well which means a finite volume shrinking into 0 volume.

First recall that

$$\frac{d}{d\tau}\det[\underline{A}] = \det[\underline{A}]\text{tr}[\frac{d\underline{A}}{d\tau}\underline{A}^{-1}]$$

Using this we can fin the material time derivative for a volume element:

$$\begin{aligned} \dot{J} &= J\text{tr}[\underline{\dot{F}}\underline{F}^{-1}] \\ \underline{\dot{F}} &= \frac{d}{dt}\left(\frac{\partial \underline{x}}{\partial \underline{X}}\right) = \frac{\partial \underline{v}}{\partial \underline{X}} = \underbrace{\frac{\partial \underline{v}}{\partial \underline{x}}}_{\underline{L}} \underbrace{\frac{\partial \underline{x}}{\partial \underline{X}}}_{\underline{F}} \rightarrow \underline{\dot{F}}\underline{F}^{-1} = \underline{L} \end{aligned}$$

$$\text{tr}[L] = L_{ii} = v_{i,i} = \text{div}[\underline{v}] \rightarrow \boxed{\dot{J} = J\text{div}[v]}$$

$$\text{if div}[v] = 0 \begin{cases} \text{for } t \in (t_1, t_2) & : \text{ motion is isochoric(volume preserving)} \\ \forall t & : \text{ incompressible motion(J=1 always), like water and rubber} \end{cases}$$

Remark: let $\underline{v} = v_i \underline{e_i} \in \mathbb{R}$ and $\underline{V} = V_A \underline{E_A} \in \mathbb{R}_0$.

$$\begin{aligned} \underline{FV} &= (F_{iA}\underline{e_i} \otimes \underline{E_A})(V_B \underline{E_B}) \\ &= F_{iA}V_B(\underbrace{\underline{E_A} \cdot \underline{E_B}}_{\delta_{AB}})\underline{e_i} \\ &= F_{iA}V_A\underline{e_i} \end{aligned}$$

Next calculation is not meaningful, this should not appear in the theoretical development:

$$\begin{aligned} \underline{Fv} &= (F_{iA}\underline{e_i} \otimes \underline{E_A})(v_j \underline{e_j}) \\ &= F_{iA}v_j(\underbrace{\underline{E_A} \cdot \underline{e_j}}_{?})\underline{e_i} \\ &= \dots \end{aligned}$$

Now $\underline{F}^T = F_{iA}\underline{E_A} \otimes \underline{e_i}$ and not $F_{Ai}\underline{e_i} \otimes \underline{E_A}$

$\underline{vFV} = \underline{F}^T\underline{vV} = (F_{Ai}\underline{E_A} \otimes \underline{e_i})(v_j \underline{e_j}) \cdot (V_B \underline{E_B})$

How about $\underline{F}^{-1}$:

$$\begin{aligned} \underline{I} &= \delta_{ij}\underline{e_i} \otimes \underline{e_j} \\ &= \frac{\partial \underline{x}}{\partial \underline{x}} \\ &= \frac{\partial \underline{x}}{\partial \underline{X}} \frac{\partial \underline{X}}{\partial \underline{x}} \\ &= \underline{FF}^{-1} \end{aligned}$$

$$\begin{aligned} \underline{F}^{-1} &= \frac{\partial \underline{X}}{\partial \underline{x}} \\ &= \frac{\partial X_A}{\partial x_i}\underline{E_A} \otimes \underline{e_i} \\ &= F_{Ai}^{-1}\underline{E_A} \otimes \underline{e_i} \end{aligned}$$

Alternatively we can check:

$$\begin{aligned} \underline{F}^{-1}\underline{F} &= (\frac{\partial X_A}{\partial x_i}\underline{E_A} \otimes \underline{e_i})(\frac{\partial x_j}{\partial X_B}\underline{e_j} \otimes \underline{E_B}) \\ &= \frac{\partial X_A}{\partial x_i}\frac{\partial x_j}{\partial X_B}\delta_{ij}\underline{E_A} \otimes \underline{E_B} \rightarrow \frac{\partial X_A}{\partial X_B} = \delta_{AB} \end{aligned}$$

2.4 Stretch and Strain

2.4.1 Stretch

$\underline{dx} = \underline{FdX}$ $\lambda = \dfrac{ds}{dS} > 0$, if stretch$>1$ then elongation, else if stretch<1 contraction. For example a rod with a length of L is stretched with ΔL. Here $\epsilon = \dfrac{\Delta L}{L}$ is engineering strain at 1D. Here also $S \in [0, L]$ and $s \in [0, L+\Delta L]$. Then $\lambda = \dfrac{\Delta s}{\Delta S} = \dfrac{L+\Delta L}{L} = 1+\epsilon$

$\lambda \underline{m} = \underline{FM} \xrightarrow{dot} \lambda^2 = \underline{FM} \cdot \underline{FM} = \underline{M} \cdot \underline{F^T FM}$ Here $\boxed{\underline{C} = \underline{F^T F}}$ is Right Cauchy-Green Deformation Tensor.

$$\underline{C} = \underline{F^T F} = (F_{iA}\underline{E_A} \otimes \underline{e_i})(F_{jB}\underline{e_j} \otimes \underline{E_B}) = \underbrace{F_{iA}F_{iB}}_{C_{AB}} \underline{E_A} \otimes \underline{E_B}$$

Here $\underline{F}$ is invertible because $J > 0$ and it admits a polar decomposition $\underline{F} = \underline{RU}$. Hence

$$\underline{F^T F} = \underline{UR^T RU} = \underline{U}^2 = \underline{C}$$

Then this right cauchy-green deformation tensor is symmetric and positive definite. This also admits the following:

$$\lambda = \sqrt{\underline{MCM}} > 0$$

Eigenvalue problem for $\underline{U} = \underline{C}^{1/2}$:

$$\underline{UV_\alpha} = \lambda_\alpha \underline{V_\alpha}$$

$$\underline{U} = \sum_\alpha \lambda_\alpha \underline{V_\alpha} \otimes \underline{V_\alpha}$$

$$\underline{C} = \sum_\alpha \lambda_\alpha^2 \underline{V_\alpha} \otimes \underline{V_\alpha}$$

Here λ_α's are principled stretches. Similarly

$$\lambda^{-1}\underline{M} = \underline{F}^{-1}\underline{m} \to \lambda^{-2} = \underline{F}^{-1}\underline{m} \cdot \underline{F}^{-1}\underline{m} = \underline{m} \cdot \underbrace{\underline{F^{-T}F^{-1}}}_{\underline{b}^{-1}} \underline{m}$$

here $\boxed{\underline{b} = \underline{FF^T}}$ is the Left Cauchy-Green Deformation Tensor.

$$\underline{b} = \underline{FF^T} = (F_{iA}\underline{e_i} \otimes \underline{E_A})(F_{jB}\underline{E_B} \otimes \underline{e_j}) = \underbrace{F_{iA}F_{jA}}_{b_{ij}} \underline{e_i} \otimes \underline{e_j}$$

$$\underline{F} = \underline{VR} \to \underline{FF^T} = \underline{VRR^T V} = \underline{V}^2 = \underline{b}$$
$$\lambda^{-1} = \sqrt{\underline{mb^{-1}m}} > 0 \quad \forall m$$

Eigenvalue problem for $\underline{V} = \underline{b}^{1/2}$:

$$\underline{V}\underline{v_\alpha} = \lambda_\alpha \underline{v_\alpha}$$

$$\underline{V} = \sum_\alpha \lambda_\alpha \underline{v_\alpha} \otimes \underline{v_\alpha}$$

$$\underline{b} = \sum_\alpha \lambda_\alpha^2 \underline{v_\alpha} \otimes \underline{v_\alpha}$$

We are going to show that the eigenvalues of U and V are shared and these are coupled with the rotation tensor R. One can argue $\underline{F} = \underline{RU} = \underline{VR} \rightarrow \underline{V} = \underline{RUR^T}$. Here $\underline{R}$ contains pure rotation but $\underline{V}$ and $\underline{U}$ are containing both stretch and rotation.

$$\underline{b} = \underline{FF^T} = \underline{RUUR^T} = \underline{RCR^T} = \sum_\alpha \lambda_\alpha^2 \underline{RV_\alpha} \otimes \underline{RV_\alpha} \rightarrow \underline{v_\alpha} = \underline{RV_\alpha}$$

So that the $\underline{R} = \underline{v_\alpha} \otimes \underline{V_\alpha}$: two point tensor. Hence:

$$\underline{F} = \underline{RU} = \sum_\alpha \lambda_\alpha \underline{v_\alpha} \otimes \underline{V_\alpha}$$

Here is the remark:

$$\begin{aligned} \underline{UV_\alpha} &= \lambda_\alpha \underline{V_\alpha} \\ \underline{RUV_\alpha} &= \lambda_\alpha \underline{RV_\alpha} \\ \underline{Vv_\alpha} &= \lambda_\alpha \underline{v_\alpha} \end{aligned}$$

2.4.2 Strain Tensors

If Strain goes to 0 then $\lambda_\alpha \rightarrow 1 \ \forall \alpha$ hence pure rotation no stretching. This strain tensor should not be rotation sensitive.

$$\begin{aligned} ds^2 - dS^2 &= ||d\underline{x}||^2 - ||d\underline{X}||^2 \\ &= \underline{FdX} \cdot \underline{FdX} - \underline{dX} \cdot \underline{I}d\underline{X} \\ &= d\underline{X}\underbrace{(\underline{C} - \underline{I})}_{2\underline{E}} d\underline{X} \end{aligned}$$

Here $\boxed{\underline{E} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}(C_{AB} - \delta_{AB})\underline{E_A} \otimes \underline{E_B}}$ is the Lagrangian(Green) Strain Tensor. Alternatively

$$\begin{aligned} ds^2 - dS^2 &= \underline{dx} \cdot \underline{dx} - \underline{dX} \cdot \underline{dX} \\ &= \underline{dx} \cdot \underline{dx} - \underline{F^{-1}dx} \cdot \underline{F^{-1}dx} \\ &= \underline{dx}\underbrace{(\underline{I} - \underline{F^{-T}F^{-1}})}_{2\underline{e}} \underline{dx} \end{aligned}$$

Here $\boxed{\underline{e} = \frac{1}{2}(\underline{I} - \underline{b}^{-1}) = \frac{1}{2}(\delta_{ij} - b_{ij}^{-1})\underline{e_i} \otimes \underline{e_j}}$ is the Eulerian(Almansi) Strain Tensor. Relation between these strain tensors are the following expression:

$$\underline{F^T e F} = \frac{1}{2}(\underline{F^T F} - \underline{I}) = \underline{E}$$

We can generalize these strain tensor:

$$\underline{E}^{(m)} = \frac{1}{m}(\underline{U}^m - \underline{I}) \text{ for } m \neq 0$$

As $m \to 0$:

$$\underline{E}^{(m)} = \ln \underline{U}$$

Where $\ln \underline{U} = \sum_\alpha \ln \lambda_\alpha \underline{V_\alpha} \otimes \underline{V_\alpha}$. $\ln \lambda = \ln \frac{ds}{dS} = \ln(1 + \epsilon)$

Chapter 3

Balance Laws

(i) Mass balance

(ii) Linear momentum balance

(iii) Angular momentum balance

(iv) Energy balance (1st law of thermodynamics)

(v) Entropy balance (2nd law of thermodynamics)

We will cover the first 3 balances in this section. Here we mention ”laws” and this means that we assume their validity.

MB+LMB+AMB→ Laws of Motion(Kinetics)

Kinetics↔Kinematics

We also assume a closed system, which means the boundary of the body is a material surface. Another remark is that the body in the reference configuration either the whole system or any free body diagram. Hence we can analyze partitions on the body, There are many boundary conditions like displacement boundary conditions(joints), force boundary conditions(forces). There is also internal forces as well.

Differential equations govern the motion but the boundary condition is the activator of this motion.

3.1 Mass Balance, Open vs Closed Systems

3.1.1 Mass Balance

$$m = \int_B dm = \begin{cases} \int_{\mathbb{R}_0} \rho_0 dV \rightarrow \rho_0 : \text{ referential mass density} \\ \int_{\mathbb{R}} \rho dv \rightarrow \rho : \text{ spatial mass density} \end{cases}$$

$$\int_{\mathbb{R}} \rho \underbrace{dv}_{JdV} = \int_{\mathbb{R}_0} \rho J dV \rightarrow \rho_0 = \rho J$$

Since we are in a closed system we conserve mass:

$$0 = \frac{dm}{dt} = \begin{cases} \frac{d}{dt}\int_{\mathbb{R}_0} \rho_0 dV(\text{global statement}) = \int_{\mathbb{R}_0} \dot{\rho}_0 dV \rightarrow \boxed{\dot{\rho}_0 = 0}(\text{local statement}) \\ \frac{d}{dt}\int_{\mathbb{R}} \rho dv(\text{global statement}) = \int_{\mathbb{R}_0} \dot{\overline{\rho J}} dV \rightarrow \boxed{\dot{\overline{\rho J}} = 0}(\text{local statement}) \end{cases}$$

Here $\dot{\overline{\rho J}} = \dot{\rho} J + \rho \dot{J} = \dot{\rho} J + \rho J \text{div}[\underline{v}] = 0 \rightarrow \dot{\rho} + \rho \text{div}[\underline{v}] = 0$

	Integral	Local
Spatial	$\frac{d}{dt}\int_{\mathbb{R}} \rho dv = 0$	$\dot{\rho} + \rho \text{div}[\underline{v}] = 0$
Referential	$\frac{d}{dt}\int_{\mathbb{R}_0} \rho_0 dV = 0$	$\dot{\rho_0} = 0$

Table 3.1: Boundary Conditions for Mass Balance in Closed Systems

3.1.2 Open vs Closed Systems

Suppose a material region, we are not analyzing the whole region bu a portion of it, this portion might be open or closed. If the boundary of the region and itself conforms with the surrounding area then it is closed however if it is not then there will be deviations in the integrals for the both parts.

$$0 = \dot{\rho} + \rho \text{div}[\underline{v}] = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \underline{x}} \cdot \underline{v} + \rho \text{div}[\underline{v}] = \frac{\partial \rho}{\partial t} + \text{div}[\rho \underline{v}]$$

Therefore: Reynolds Transport Theorem(RTT) for a closed system is:

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{R}} \rho dv \\ &= \int_{\mathbb{R}} (\dot{\rho} + \rho \text{div}[\underline{v}]) dv \\ &= \int_{\mathbb{R}} (\frac{\partial \rho}{\partial t} + \text{div}[\rho \underline{v}]) dv \\ &= \underbrace{\int_{\mathbb{R}} \frac{\partial \rho}{\partial t} dv}_{\substack{\text{volume contains} \\ \text{raw distrbution} \\ \text{that chages with time}}} + \underbrace{\int_{\partial \mathbb{R}} \rho \underline{v} \cdot \underline{n} da}_{\substack{\text{Volume attempts to} \\ \text{encompass } \rho \text{ at veloscity } \underline{v} \\ \text{which happens to be} \\ \text{material velocity}}} \end{aligned} \tag{3.1}$$

Hence the rate of change of mass in a volume whose boundary conforms(closed) to the material velocity.

Now we are going to tackle the other scenario where the boundary is not going to conform to the motion of the particles and there will be discrepancy between domains.

In general RTT for an open system, be aware we are taking the material time derivative with the following operation:

$$0 \neq \underbrace{\frac{D}{Dt}\int_{\mathbb{R}} \rho dv}_{\substack{\text{Whose boundary moves} \\ \text{at an indepent(open)} \\ \text{velocity to engulf} \\ \text{more/less material}}} = \int_{\mathbb{R}} \frac{\partial \rho}{\partial t} dv + \int_{\partial \mathbb{R}} \rho \underline{w} \cdot \underline{n} da \tag{3.2}$$

This independent motion is embodied by the $\underline{w}$, if it is equal to material velocity then this equation would be 0 hence the closed system. Here if we substract the equation 3.2 from the 3.1, this delivers the following expression:

$$0 = \frac{d}{dt}\int_{\mathbb{R}} \rho d = \frac{D}{Dt}\int_{\mathbb{R}} \rho dv + \int_{\partial \mathbb{R}} \rho(\underline{v} - \underline{w}) \cdot \underline{n} da$$

Or

$$\frac{D}{Dt}\int_{\mathbb{R}} \rho dv \overbrace{=}^{(MB)} \int_{\partial \mathbb{R}} \rho(\underline{w} - \underline{v}) \cdot \underline{n} da$$

For instance let $\underline{w} = 0 : -\underline{v} \cdot \underline{n} > 0$ mass influx, $-\underline{v} \cdot \underline{n} < 0$ mass efflux.
$\underline{v} = 0 : \underline{v} \cdot \underline{n} > 0$ volume expand to engulf more material.

3.2 Linear and Angular Momentum Balance

For the mass conservation we introduced the following set of equations:

$$\frac{d}{dt} m = \frac{d}{dt}\int db = \frac{d}{dt}\int_{\mathbb{R}} \rho dv = \frac{d}{dt}\int_{\mathbb{R}_0} \rho J dV \rightarrow \dot{\rho} + \rho \text{div}[\underline{v}] = 0$$

Linear(Translational) Momentum of the material in $\mathbb{R}$:

$$\underline{P} = \int dP = \int_{\mathbb{R}} \underbrace{p}_{\rho \underline{v}} dv = \int_{\mathbb{R}} \rho \underline{v} dv$$

Angular(Rotational/Moment of) momentum of the material in $\mathbb{R}$ with respect to a stationary $\underline{x_0}$(not necessarily the origin), here material point is at $\underline{x}$ and the difference between is $\underline{r_0} = \underline{x} - \underline{x_0}$:

$$\begin{aligned} \mathcal{H}^0 &= \int \underline{r_0} \times \underline{dP} \\ &= \int_{\mathbb{R}} \underline{r_0} \times \rho \underline{v} dv \end{aligned}$$

Laws of motion are separated into two forms:

(1) Integral/Eulerian

(2) Local(Differential)/Cauchy

3.2.1 Euler's Laws of Motion

There exists a (inertial/Newtonian) frame of reference such that $\underline{\dot{\mathcal{P}}} = \underline{F}$(LMB) and $\underline{\dot{\mathcal{H}}^0} = \underline{M}^0$(AMB). Here $\underline{F}$ is the net(resultant) force and $\underline{M}^0$ is net moment about $\underline{x_0}$. These forces and moments can be decomposed as the body and the surface forces/moments:

$$\begin{aligned} \underline{F} &= \underline{F}_B + \underline{F}_S \\ \underline{M}^0 &= \underline{M}_B^0 + \underline{M}_S^0 \end{aligned}$$

Lets further inspect these decompositions:

$$\begin{aligned} \underline{F}_B &= \int dF_B \\ &= \int_{\mathbb{R}} f_B dv \\ &= \int_{\mathbb{R}} \rho \underline{b} dv \quad \underline{b}: \text{ body force per unit mass} \\ \underline{F}_S &= \int_{\partial\mathbb{R}} \underline{t} da \quad \underline{t}: \text{ traction} \end{aligned}$$

$\underline{F}_B$: gravity, electromagnetic forces etc
$\underline{F}_S$: due to contact with a surface

$$\begin{aligned} \underline{M}_B^0 &= \int \underline{r_0} \times d\underline{F}_B \\ &= \int_{\mathbb{R}} \underline{r_0} \times \rho \underline{b} dv \\ \underline{M}_S^0 &= \int \underline{r_0} \times d\underline{F}_S \\ &= \int_{\partial\mathbb{R}} \underline{r_0} \times \rho \underline{t} da \end{aligned}$$

So coupling the moments with the the forces will introduce some consequences, this means that we are dealing with a non-polar medium, if we were to define new variables as in the case of forces we could define polar medium. Non-polar medium, thus the stress tensor is symmetric.

With all the assumptions and, we can restate the following. Integral/spatial forms of the Euler's laws of motion:

$$\begin{aligned} \text{LMB}: \frac{d}{dt}\int_{\mathbb{R}} \rho \underline{v} dv &= \int_{\mathbb{R}} \rho \underline{n} dv + \int_{\partial\mathbb{R}} \underline{t} da \\ \text{AMB}: \frac{d}{dt}\int_{\mathbb{R}} \underline{r_0} \times \rho \underline{v} dv &= \int_{\mathbb{R}} \underline{r_0} \times \rho \underline{b} dv + \int_{\partial\mathbb{R}} \underline{r_0} \times \underline{t} da \end{aligned}$$

If $\underline{F}/\underline{M}^0 = 0$ then $\underline{\dot{\mathcal{P}}}/\underline{\dot{\mathcal{H}}^0} = 0 \rightarrow$ conservation law

Returning back to LMB, where $\tilde{\underline{t}}(\underline{x}, t, \underline{t}) = \tilde{\underline{T}}(\underline{x}, t)\underline{n}$ is the cauchy stress tensor:

$$\frac{d}{dt}\int_{\mathbb{R}} \rho dv = \int_{\mathbb{R}} \rho \underline{b} dv + \int_{\partial\mathbb{R}} \underline{t} da$$
$$\int_{\mathbb{R}_0} \rho_0 \dot{\underline{v}} dV = \int_{\mathbb{R}} \rho \underline{b} dv + \int_{\mathbb{R}} \text{div}[\underline{T}] dv$$
$$\int_{\mathbb{R}} \rho \dot{\underline{v}} dv = \int_{\mathbb{R}} \rho \underline{b} dv + \int_{\mathbb{R}} \text{div}[\underline{T}] dv$$

Hence

$$\int_{\mathbb{R}} (\rho\dot{\underline{v}} - \rho\underline{b} - \text{div}[\underline{T}]) dv = 0 \rightarrow \text{div}[\underline{T}] + \rho\underline{b} = \rho\dot{\underline{v}} (\text{ local/spatial form of LMB})$$

Returning to AMB: $\underline{T}^T = \underline{T}$ local form of AMB.
To summarize, spatial form of LMB/AMB:

	Euler(Integral)	Cauchy(Local)
LMB	$\int_{\partial\mathbb{R}} \underline{t} da + \int_{\mathbb{R}} \rho \underline{b} dv = \frac{d}{dt}\int_{\mathbb{R}} \rho \underline{v} dv$	$\text{div}[\underline{T}] + \rho\underline{b} = \rho\dot{\underline{v}}$
AMB	$\int_{\partial\mathbb{R}} \underline{r_0} \times \underline{t} da + \int_{\mathbb{R}} \underline{r_0} \times \rho \underline{b} dv = \frac{d}{dt}\int_{\mathbb{R}} \underline{r_0} \times \rho \underline{v} dv$	$\underline{T}^T = \underline{T}$

Table 3.2: LMB and AMB Laws

3.3 Symmetry of the Cauchy Stress Tensor and Cauchy's Theorem

Proof of the $\underline{T}^T = \underline{T}$, $\underline{r_0} = \underline{x} - \underline{x_0}$:

$$(\text{AMB}:)\ \underbrace{\frac{d}{dt}\int_{\mathbb{R}} \underline{r_0} \times \rho\underline{v}dv}_{(I)} = \int_{\mathbb{R}} \underline{r_0} \times \rho\underline{b}dv + \underbrace{\int_{\partial\mathbb{R}} \underline{r_0} \times \underline{t}da}_{(II)}$$

$$\begin{aligned}
(I) &= \frac{d}{dt}\int_{\mathbb{R}_0} \underline{r_0} \times \rho_0\underline{v} = \int_{\mathbb{R}_0} (\dot{\underline{r_0}} \times \rho_0\underline{v} + \underline{r_0} \times \rho_0\dot{\underline{v}})dv \\
&= \int_{\mathbb{R}_0} \underline{r_0} \times \rho_0\dot{\underline{v}}dv \\
&= \int_{\mathbb{R}} \underline{r_0} \times \rho\dot{\underline{v}}dv \\
(II) &= \int_{\partial\mathbb{R}} \underbrace{\underline{r_0} \times \underline{Tn}}_{e_{ijk}(T_{kl}n_l)\underline{e}_i}\ da \\
&= \int_{\mathbb{R}} (e_{ijk}r_{0j}T_{kl})_{,l}\underline{e}_i dv \\
&= \int_{\mathbb{R}} e_{ijk}(r_{0j,l}T_{kl} + r_{0j}T_{kl,l})\underline{e}_i dv \\
&= \int_{\mathbb{R}} e_{ijk}(\frac{\partial x_j - x_{0j}}{\partial x_l}T_{kl} + r_{0j}T_{kl,l})\underline{e}_i dv \\
&= \int_{\mathbb{R}} e_{ijk}(\frac{\partial x_j}{\partial x_l}T_{kl} + r_{0j}T_{kl,l})\underline{e}_i dv \\
&= \int_{\mathbb{R}} e_{ijk}(\delta_{jl}T_{kl} + r_{0j}T_{kl,l})\underline{e}_i dv \\
&= \int_{\mathbb{R}} (e_{ijk}T_{kj}\underline{e}_i + \underline{r_0} \times \text{div}[\underline{T}])dv
\end{aligned}$$

$$(\text{AMB}:) \int_{\mathbb{R}} \underline{r_0} \times \underbrace{(\rho\dot{\underline{v}} - \rho\underline{b} - \text{div}[\underline{T}])}_{=0 \text{ since LMB(Local)}} = \int_{\mathbb{R}} e_{ijk}T_{kj}\underline{e}_i dv \rightarrow e_{ijk}T_{kj} = 0$$

Now using the $e - \delta$ identity:

$$\begin{aligned}
0 &= e_{imn}e_{ijk}T_{kj} \\
&= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})T_{kj} \\
&= T_{nm} - T_{mn} \rightarrow T_{mn} = T_{nm} : \text{(symmetry)}
\end{aligned}$$

3.3.1 Stress Tensor

All quantities depend on $\{\underline{x}, t\}$, but traction $\underline{t}$ depends more. Lets have a bar and we are going to pull the bar from both sides with the force P, to analyze the bar I am going to divide it into two from the position $\underline{x}$ at time t and look at the pieces, For the free body diagrams of this pieces we have a traction field and a force acting on the body. Each body piece has a traction field, let the left piece have the $\underline{t}^{(1)}(\underline{x}, t)$ and the right is $\underline{t}^{(2)}(\underline{x}, t)$. Then the traction should be also depend on the outward unit normal as well since these two fields should be different hence for the eulerian representations:

(1) $\tilde{\underline{t}}(\underline{x}, t, \underline{n})$

(2) $\underline{t}^{(1)} = \tilde{\underline{t}}(\underline{x}, t, \underline{n}) = -\tilde{\underline{t}}(\underline{x}, t, -\underline{n})$

So each traction interface should have a counterpart, this is called the Cauchy's Lemma:

$$\underbrace{\tilde{\underline{T}}(\underline{x}, t)\underline{n}}_{\text{Cauchy's Theorem}} = \underbrace{\boxed{\tilde{\underline{t}}(\underline{x}, t, \underline{n}) = -\tilde{\underline{t}}(\underline{x}, t, -\underline{n})}}_{\text{Cauchy's Lemma}}$$

Now we are goinf to show the equivalence of these equations

3.3.2 Cauchy Process

Time plays a big role but time plays no role in the derivation of this process, we will omit it for simplification. Now, lets choose a tetrahedron sitting at $\underline{x}$ and choose the tetrahedron to be formed by the $\underline{e_1}, \underline{e_2}, \underline{e_3}$. Let this be the Cauchy Tetrahedron $\mathcal{C}$. Now this tetrahedron has 4 faces which has the outward unit normals with the areas of $-\underline{e_1}(A_1), -\underline{e_2}(A_2), -\underline{e_3}(A_3), \underline{n}(A)$. And let the inclined surface have the distance h to the other vertex. Lets recall some geometric results:

$$\underline{A} = A\underline{n} = \underbrace{An_i}_{A_i}\underline{e}_i$$

$$V = \frac{1}{3}Ah$$

Lets also define a position vector that resides inside this tetrahedron $\underline{y}$.

$$\text{(LMB :)} \int_{\mathcal{C}} \rho\dot{\underline{v}}dv = \int_{\mathcal{C}} \rho\underline{b}dv + \int_{\partial\mathcal{C}} \underline{t}da \text{ now we ared decompose the surface integral into faces}$$

$$\int_{\mathcal{C}} \rho(\dot{\underline{v}} - \underline{b})dv = \int_{\partial\mathcal{C}^{(n)}} \tilde{\underline{t}}(\underline{y}, \underline{n})da + \sum_i \int_{\partial\mathcal{C}^{(i)}} \tilde{\underline{t}}(\underline{y}, -\underline{e}_i)da$$

$$(*) \int_{\mathcal{C}} \tilde{f}(\underline{y})dv = \int_{\partial\mathcal{C}^{(n)}} \tilde{\underline{t}}(\underline{y}, \underline{n})da - \sum_i \int_{\partial\mathcal{C}^{(i)}} \tilde{\underline{t}}(\underline{y}, \underline{e}_i)da$$

Now recall the mean value theorem for any function $g(y)$ at a point $\bar{y}$ and integration length of L, for this thm we assume smoothness and continuous requirement:

$$\frac{1}{L}\int g dy = g(\bar{y})$$

Now, assuming the sufficient smoothness use the mean value theorem:

$$\exists \underline{\bar{y}} \in \mathcal{D} : \int_{\mathcal{D}} \tilde{g}(\underline{y}) dv = \tilde{g}(\underline{\bar{y}})|\mathcal{D}|$$
$$\exists \underline{\bar{y}} \in \partial\mathcal{D} : \int_{\partial\mathcal{D}} \tilde{g}(\underline{y}) dv = \tilde{g}(\underline{\bar{y}})|\partial\mathcal{D}|$$

Hence $(*)$ is equivalent to:

$$\tilde{f}(\underline{\bar{y}})V = \underline{\tilde{t}}(\underline{\bar{y}}^{(n)}, \underline{n})A - \sum_i \underline{\tilde{t}}(\underline{\bar{y}}^{(i)}, \underline{e}_i)A_i$$

Then we have 5 special points that satisfies this equation, we dont care about them only thing is that they exists:

$$\tilde{f}(\underline{\bar{y}})\frac{1}{3}Ah = \underline{\tilde{t}}(\underline{\bar{y}}^{(n)}, \underline{n})A - \sum_i \underline{\tilde{t}}(\underline{\bar{y}}^{(i)}, \underline{e}_i)An_i$$
$$\underline{\tilde{t}}(\underline{\bar{y}}^{(n)}, \underline{n}) - \sum_i \underline{\tilde{t}}(\underline{\bar{y}}^{(i)}, \underline{e}_i)n_i = \tilde{f}(\underline{\bar{y}})\frac{1}{3}h$$

Let $\mathcal{C}$ shrink towards the point $\underline{x} : h \to 0, \ \underline{\bar{y}}^{(n)}, \underline{\bar{y}}^{(i)} \to \underline{x}$:

$$\underline{\tilde{t}}(\underline{x}, \underline{n}) = \sum_i \underbrace{\underline{\tilde{t}}(\underline{x}, \underline{e}_i)}_{\underline{\tilde{t}}^{(i)}(\underline{x})} n_i = \underline{\tilde{t}}^{(i)} \underbrace{n_i}_{\underline{e}_i \cdot \underline{n}} = [\underline{\tilde{t}}^{(i)} \otimes \underline{e}_i]\underline{n}$$

In general, $\boxed{\underline{\tilde{t}}(\underline{x}, t, \underline{n}) = \underline{\tilde{T}}(\underline{x}, t)\underline{n}}$

Recall undergraduate mechanics, for a body there are stresses in the direction of the normals $\sigma_x, \sigma_y, \sigma_z$ and the shear stresses on the face along the other axes $\tau_{zy} = \tau_{yz}, \tau_{xy} = \tau_{yx}, \tau_{zx} = \tau_{xz}$. Matrix representation of the stress tensor is:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Or for the 3 face $\underline{T} = \underline{t^{(i)}} \otimes \underline{e_i}$:

$$[t^{(1)}] = \begin{bmatrix} \sigma_x \\ \tau_{yx} \\ \tau_{zx} \end{bmatrix}, \ [t^{(2)}] = \begin{bmatrix} \tau_{xy} \\ \sigma_y \\ \tau_{zy} \end{bmatrix}, \ [t^{(3)}] = \begin{bmatrix} \tau_{xz} \\ \tau_{zy} \\ \sigma_z \end{bmatrix} \to [\sigma] \overbrace{=}^{\text{if deformations are small}} [\{t^{(1)}\}\{t^{(2)}\}\{t^{(3)}\}] = [T]$$

If the deformations are large there are other stress tensors that gets in the way.

$$\underline{t} = \underline{Tn} = \underline{Te_i} = (\underline{t^{(j)}} \otimes \underline{e_j})\underline{e_i} = \underline{t}^{(j)}\delta_{ij} = \underline{t}^{(i)}$$
$$T_{ij} = \underline{e_i} \cdot \underline{Te_j} = \underline{e_i} \cdot \underline{t^{(j)}} = t_i^{(j)}, \text{j: surface normal, i: component} \to \sigma_{ij} = \sigma ji$$

3.4 Referential Forms of Linear and Angular Momentum Balance

For an arbitrary surface with normal $\underline{n}$ and stress components as $\sigma_n \underline{n}$ as the normal stress and $\underline{\tau}$ as the tangential part:

$$\underline{t} = \sigma_n \underline{n} + \underline{\tau}$$

$$\sigma_n = \underline{n} \cdot \underline{t} = \underline{n} \cdot \underline{Tn}, \quad \underline{\tau} = \underline{t} - \sigma_n \underline{n}$$

Particular tyoes of stress states:

(1) Spherical(pure pressure): $\underline{T} = -p\underline{I} \rightarrow \underline{t} = \underline{Tn} = -p\underline{n} \forall n$

$$[T] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

(2) Uniaxial/Biaxial/Triaxial Tension/Compression:

$$[T] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

(3) Simple shear

$$[T] = \begin{bmatrix} 0 & s_1 & s_2 \\ s_1 & 0 & s_3 \\ s_2 & s_3 & 0 \end{bmatrix}$$

3.4.1 Referential Forms of Linear and Angular Momentum Balance

Lets have a bar and we are pulling it from both sides with F and we have a reference configuration $\mathbb{R}_0$ and a cross section on that A. This are becomes a after the deformation on the current configuration $\mathbb{R}$. True Stress:

$$\tilde{\sigma} = \frac{F}{a}$$

Engineering stress:

$$\sigma = \frac{F}{A} = \frac{F}{a}\frac{a}{A} = \tilde{\sigma}\frac{a}{A}$$

Knowledge of deformation allows relating σ to $\tilde{\sigma}$. Lets define $\hat{\sigma}$:

$$\hat{\sigma} = \tilde{\sigma}\left(\frac{a}{A}\right)^2$$

We could build other stress measures without a physical meaning like we did just now. Now, lets have a body on reference config $\mathbb{R}_0$ and current config $\mathbb{R}$, also a motion map $\underline{\chi}$. Lets have a cross section on the reference with differential area dA with normal $\underline{N}$ and counterparts on the current body as da and $\underline{n}$, Nanson's Formula dictates:

$$\underline{n}da = \underbrace{\underline{F}^{\#}}_{J\underline{F}^{-T}} \underline{N}d\underline{A}$$

Also the traction relation:

$$\tilde{\underline{t}}(\underline{x}, t, \underline{n}) = \tilde{\underline{T}}(\underline{x}, t)\underline{n}$$

From these attributes we can define the infinitesimal force:

$$\begin{aligned} d\underline{F} &= \underline{t}da = \underline{Tn}da \\ &= \underline{p}dA : \begin{cases} \text{still acts on } \mathbb{R} \text{ in the direction of } \underline{t} \\ \underline{t} = t_i\underline{e}_i \\ \underline{p} = p_i\underline{e}_i \end{cases} \qquad \rightarrow \underline{p} = \underline{t}\frac{da}{dA} \end{aligned}$$

Construct a new stress tensor:

$$\begin{aligned} \underline{t}da &= \underline{Tn}da \\ &= \underline{TF}^{\#}\underline{N}dA \\ &= \underline{p}dA \rightarrow \underline{p} = \underline{TF}^{\#}\underline{N} \end{aligned}$$

Then the spatial form $\hat{\underline{p}}$ and referential form $\underline{N}$ is coupled with the two point tensor $\hat{\underline{P}}$

$$\hat{\underline{p}}(\underline{X}, t, \underline{N}) = \hat{\underline{P}}(\underline{X}, t)\underline{N}, \quad \underline{P} = P_{iA}\underline{e_i} \otimes \underline{e_A}$$

Now we have $\underline{t}$: Cauchy traction, $\underline{T}$: Cauchy Stress, $\underline{p}$: Piola traction and $\underline{P}$: 1st Piola-Kuchhoff stress(1st P-K)

$$\underline{P} = J\underline{TF}^{-T} \rightarrow \underline{T} = \frac{1}{J}\underline{PF}^{T}$$

AMB is easy:

$$\underline{T} = \frac{1}{J}\underline{PF}^{T} = \underline{T}^{T} = \frac{1}{J}\underline{FP}^{T} \rightarrow \underline{PF}^{T} = \underline{FP}^{T}(\text{ local/referntial form of AMB})$$

For LMB we have the following:

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} \rho\underline{v}dv &= \int_{\mathbb{R}} \rho\underline{b}dv + \int_{\partial\mathbb{R}} \underline{t}da \\ \frac{d}{dt}\int_{\mathbb{R}_0} \rho_0\underline{v}dV &= \int_{\mathbb{R}} \rho\underline{b}dv + \int_{\partial\mathbb{R}_0} \underbrace{\underline{p}}_{\underline{PN}} dA \\ \int_{\mathbb{R}_0} \rho_0\dot{\underline{v}}dV &= \int_{\mathbb{R}_0} \rho_o\underline{b}dv + \int_{\mathbb{R}_0} \text{Div}[\underline{P}]da \end{aligned}$$

Hence the local form of LMB in the referential space is:

$$\text{Div}[\underline{P}] + \rho_0 \underline{b} = \rho_0 \dot{\underline{v}}$$

To summarize, the local forms of the balance laws are:

	Spatial	Referential
MB	$\dot{\rho} + \rho \text{div}[\underline{v}] = 0$	$\rho_0 = \rho J$
LMB	$\text{div}[\underline{T}] + \rho \underline{b} = \rho \dot{\underline{v}}$	$\text{Div}[\underline{P}] + \rho_0 \underline{b} = \rho_0 \dot{\underline{v}}$
AMB	$\underline{T}^T = \underline{T}$	$\underline{PF}^T = \underline{FP}^T$

Table 3.3: Summary of Balance Laws

These are all subject to the appropriate initial/boundary conditions. Concentrate on the spatial form and assume $\underline{T}$ is symmetric, lets see the parameters:

$$\begin{cases} \#\text{ of equations:} \hat{\underline{\chi}}(\underline{X}, t)(3) + \rho(1) + \underline{T}(6) = 10 \\ \#\text{ of equations:} MB(1) + LMB(3) = 4 \end{cases}$$

We have 6 missing equations. We must relate kinematics to kinetics. Kinematics relate to strain and kinetics relate to stress. A one way to do it, as an undergraduate mechanics way is generalized Hooke's law, introducing a stiffness matrix:

$$\{\sigma\}_{6x1} = [C]_{6x6}\{t\}_{6x1}$$

These equations can change among different materials. One can introduce other stress measures:

$$\underline{\tau} = J\underline{T} : \text{ Kirchoff Stress Tensor}$$

So that

$$\int_{\mathbb{R}} \underline{T} dv = \int_{\mathbb{R}_0} \underline{\tau} dV$$

Or

$$\underline{S} = \underline{F}^{-1}\underline{P} : \text{ 2nd P-K Stress Tensor}$$

so that

$$\underline{S} = \underline{S}^T \quad \underline{S} = J\underline{F}^{-1}\underline{TF}^{-T} \rightarrow \underline{T} = \frac{1}{J}\underline{FST}^T$$

Chapter 4

Rigid Body Dynamics

4.1 RBD I

Lets have a body on reference configuration $\mathbb{R}_0$ at $\underline{X}$ and its current configuration $\mathbb{R}$ at $\underline{x}$. In general, center of mass CM:

$$CM = \begin{cases} \underline{\bar{X}} = \frac{1}{m}\int_{\mathbb{R}_0} \rho_0 \underline{X} dV \\ \underline{\bar{x}} = \frac{1}{m}\int_{\mathbb{R}} \rho \underline{x} dV \end{cases}$$

Relative position vector:

$$\begin{cases} \text{wrt point } o: \underline{r_0} = \underline{x} - \underline{x_0} \\ \text{wrt CM:} \underline{\bar{r}} = \underline{x} - \underline{\bar{x}} \\ \underline{\bar{R}} = \underline{X} - \underline{\bar{X}} \end{cases}$$

Now we should define rigid body motion:

$$\underline{x}(\underline{X}, t) = \underbrace{\underline{Q}(t)}_{\text{rotation}} \underline{X} + \underbrace{\underline{c}(t)}_{\text{translation}}$$

$$\underline{Q}: \text{ proper orthogonal}$$

$$\underline{F} = \underline{Q} \rightarrow \det[\underline{F}] = 1 \rightarrow \rho_0 = \rho$$

(1) $\underline{\bar{r}} = \underline{Q}\underline{\bar{R}}$ holds for rigid body dynamic(in general this not holds)

$$\begin{aligned} \underline{\bar{r}} &= \underline{x} - \underline{\bar{x}} \\ &= \underline{Q}\underline{X} + \underline{c} - \frac{1}{m}\int_{\mathbb{R}} \rho(\underline{QX} + c)dv \\ &= \underline{QX} - \frac{1}{m}\int_{\mathbb{R}} \rho\underline{QX}dv + \underline{c} - \frac{1}{m}\int_{\mathbb{R}} \rho\underline{c}dv \\ &= \underline{QX} - \underline{Q}\frac{1}{m}\int_{\mathbb{R}_{\not{0}}} \rho_0\underline{X}dV + \underline{c} - \underline{c} \\ &= \underline{QX} - \underline{Q\bar{X}} = \underline{Q}(\underline{X} - \underline{\bar{X}}) = \underline{Q\bar{R}} \end{aligned}$$

(2) This item is always true $\underline{P} = m\underline{\bar{v}}$ where $\underline{\bar{v}} = \underline{\dot{\bar{x}}}$. It follows that $\underline{F} = m\underline{\dot{\bar{v}}}$ by LMB $\underline{\dot{P}} = \underline{F}$.

$$\underline{P} = \int_{\mathbb{R}} \rho \underline{v} dv = \frac{d}{dt} \int_{\mathbb{R}} \rho \underline{x} dv = \frac{d}{dt} \frac{m}{m} \int_{\mathbb{R}} \rho \underline{x} dv = m\underline{\bar{v}}$$

(3) This item is also always true. Originally, $\underline{M}^0 = \underline{\dot{\mathcal{H}}}^0$, this item dictates that $\underline{M}^{CM} = \underline{\dot{\mathcal{H}}}^{CM}$, i.e. AMB holds with respect to center of mass.

$$\begin{aligned}
\underline{\mathcal{H}}^0 &= \int_{\mathbb{R}} \underline{r}_0 \times \rho \underline{v} dv \\
&= \int_{\mathbb{R}} (\underline{\bar{r}} + \underline{\bar{x}} - \underline{x}_0) \times \rho \underline{v} dv \\
&= (\underline{\bar{x}} - \underline{x}_0) \times \underbrace{\int_{\mathbb{R}} \rho \underline{v} dv}_{\underline{P}} + \underbrace{\int_{\mathbb{R}} \underline{\bar{r}} \times \rho \underline{v} dv}_{\underline{\mathcal{H}}^{CM}}
\end{aligned}$$

Likewise

$$\begin{aligned}
\underline{M}^0 &= \int_{\mathbb{R}} \underline{r}_0 \times \rho \underline{b} dv + \int_{\partial\mathbb{R}} \underline{r}_0 \times \underline{t} da \\
&= \int_{\mathbb{R}} (\underline{\bar{r}} + \underline{\bar{x}} - \underline{x}_0) \times \rho \underline{b} dv + \int_{\partial\mathbb{R}} (\underline{\bar{r}} + \underline{\bar{x}} - \underline{x}_0) \times \underline{t} da \\
&= (\underline{\bar{x}} - \underline{x}_0) \times \underbrace{\left(\int_{\mathbb{R}} \rho \underline{b} dv + \int_{\partial\mathbb{R}} \underline{t} da \right)}_{\underline{F}} + \underbrace{\int_{\mathbb{R}} \underline{\bar{r}} \times \rho \underline{b} dv + \int_{\partial\mathbb{R}} \underline{\bar{r}} \times \underline{t} da}_{\underline{M}^{CM}}
\end{aligned}$$

Now

$$\begin{aligned}
\underline{\dot{\mathcal{H}}}^0 &= \underbrace{\underline{\dot{\bar{x}}} \times \underline{P}}_{0} + (\underline{\bar{x}} - \underline{x}_0) \times \underbrace{\underline{\dot{P}}}_{\underline{F}} + \underline{\dot{\mathcal{H}}}^{CM} \\
&= (\underline{\bar{x}} - \underline{x}_0) \times \underline{F} + \underline{\dot{\mathcal{H}}}^{CM}
\end{aligned}$$

and

$$\underline{M}^0 = (\underline{\bar{x}} - \underline{x}_0) \times \underline{F} + \underline{M}^{CM} \rightarrow \underline{M}^{CM} = \underline{\dot{\mathcal{H}}}^{CM}$$

(4) One may write $\underline{\dot{\bar{r}}} = \underline{\omega} \times \underline{\bar{r}}$ where ω : angular velocity vector(constant for $\mathbb{R}$):

$$\begin{aligned}
\underline{\bar{r}} &= \underline{Q}\underline{\bar{R}} \qquad \underline{\bar{R}} = \underline{Q}^T \underline{\bar{r}} \\
\underline{\dot{\bar{r}}} &= \underline{\dot{Q}}\underline{\bar{R}} \\
&= \underline{\dot{Q}Q^T}\underline{\bar{r}} \qquad \underline{\dot{\overline{QQ^T}}} = \underline{0} = \underline{\dot{Q}Q^T} + \underline{Q\dot{Q}^T} \rightarrow \underline{\dot{Q}Q^T} = -(\underline{\dot{Q}Q^T})^T (\text{skew} : \Omega, \ \text{axial vector:}\omega) \\
&= \underline{\omega} \times \underline{\bar{r}}
\end{aligned}$$

(5) $\underline{\mathcal{H}}^{CM} = \underline{J}^{CM}\underline{\omega}$ where $\underline{J}^{CM}$: inertia tensor with respect to CM:

$$\begin{aligned}
\underline{\underline{\mathcal{H}}^{CM}} &= \int_{\mathbb{R}} \underline{\bar{r}} \times \rho \underline{v} dv \\
&= \int_{\mathbb{R}} \underline{\bar{r}} \times \rho \underline{\dot{\bar{r}}} dv + \underbrace{\left(\int_{\mathbb{R}} \rho \underline{\bar{r}} dv \right)}_{\underline{0}} \times \underline{\bar{v}} \\
&= \int_{\mathbb{R}} \rho \underline{\bar{r}} \times (\underline{\omega} \times \underline{\bar{r}}) dv \\
&\text{(Recall:)} \underline{a} \times (\underline{b} \times \underline{c}) = ((\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}) \\
&= \int_{\mathbb{R}} \rho((\underline{\bar{r}} \cdot \underline{\bar{r}})\underline{\omega} - (\underline{\bar{r}} \cdot \underline{\omega})\underline{\bar{r}}) dv \\
&= \underbrace{\int_{\mathbb{R}} \rho(\underline{\bar{r}} \cdot \underline{\bar{r}} \underline{I} - \underline{\bar{r}} \otimes \underline{\bar{r}}) dv}_{\underline{J}^{CM}} \underline{\omega}
\end{aligned}$$

4.2 RBD II

(6) $\underline{J}^{CM} = \underline{Q}\underline{J}_0^{CM}\underline{Q}^T$:

$$\begin{aligned}
\underline{J}^{CM} &= \int_{\mathbb{R}} \rho(\underline{\bar{r}} \cdot \underline{\bar{r}} \underline{I} - \underline{\bar{r}} \otimes \underline{\bar{r}}) dv & (\underline{\bar{r}} = \underline{Q}\underline{\bar{R}}) \\
&= \int_{\mathbb{R}_0} \rho_0(\underline{\bar{R}} \cdot \underline{\bar{R}} \underline{Q}\underline{Q}^T - \underline{Q}(\underline{\bar{R}} \otimes \underline{\bar{R}})\underline{Q}^T) dV \\
&= \underline{Q} \int_{\mathbb{R}_0} \rho_0(\underline{\bar{R}} \cdot \underline{\bar{R}} \underline{I} - \underline{\bar{R}} \otimes \underline{\bar{R}}) dV \underline{Q}^T \\
&= \underline{Q}\underline{J}_0^{CM}\underline{Q}^T
\end{aligned}$$

Summary:

$$\underline{x}(\underline{X}, t) = \underline{Q}(t)\underline{X} + \underline{c}(t) \rightarrow \text{ what are Q and c?}$$
$$(LMB :)\underline{F} = m\underline{\dot{\bar{v}}}$$

From here solve for $\underline{\bar{a}} = \underline{\dot{\bar{v}}}$ then update $\underline{\bar{v}}$:

$$\underline{\bar{v}} = \underline{v}(\underline{\bar{X}}) = \underline{\dot{Q}}\underline{\bar{X}} + \underline{\dot{c}} \tag{4.1}$$

$$\begin{aligned}
(AMB :)\underline{M}^{CM} &= \underline{\dot{\mathcal{H}}}^{CM} \leftarrow \underline{\mathcal{H}}^{CM} = \underline{J}^{CM}\underline{\omega} \leftarrow \underline{\dot{\bar{r}}} = \underline{\omega} \times \underline{\bar{r}} = \underline{\Omega \bar{r}} \\
&= \underline{\dot{J}}^{CM}\underline{\omega} + \underline{J}^{CM} \underbrace{\underline{\dot{\omega}}}_{\underline{\alpha}} \\
&= \underbrace{\underline{\omega} \times \underline{J}^{CM}}_{\text{TBS in HW}} \underline{\omega} + \underline{J}^{CM}\underline{\alpha}
\end{aligned}$$

From here solve for $\underline{\alpha}$ then update $\underline{\omega}$

(7) Prove König decomposition for a kinetic energy T:

$$T = T_{\text{traslational}} + T_{\text{rotational}}$$

$$\begin{aligned}
T &= \frac{1}{2}\int_{\mathbb{R}} \rho \underline{v}\cdot\underline{v} dv \\
&= \frac{1}{2}\int_{\mathbb{R}} \rho(\dot{\underline{\bar{r}}} + \underline{\bar{v}})\cdot(\dot{\underline{\bar{r}}} + \underline{\bar{v}}) dv \\
&= \frac{1}{2}\underbrace{\int_{\mathbb{R}} \rho dv}_{m}\, \underline{\bar{v}}\cdot\underline{\bar{v}} + \underline{\bar{v}} \underbrace{\int_{\mathbb{R}} \rho\dot{\underline{\bar{r}}} dv}_{\frac{d}{dt}\int_{\mathbb{R}}\rho\underline{\bar{r}}dv=0} + \frac{1}{2}\underbrace{\int_{\mathbb{R}} \rho\dot{\underline{\bar{r}}}\cdot\dot{\underline{\bar{r}}}\, dv}_{(*)\dot{\underline{\bar{r}}}=\underline{\omega}\times\underline{\bar{r}}}
\end{aligned}$$

$$\begin{aligned}
(*) &= \underbrace{e_{ijk}e_{imn}}_{\delta_{jm}\delta_{km}-\delta_{jn}\delta_{kn}} \quad \omega_j\bar{r}_k\omega_m\bar{r}_n \\
&= (\omega_m\omega_m)(\bar{r}_n\bar{r}_n) - \omega_m(\bar{r}_m\bar{r}_n)\omega_n \\
&= \underline{\omega}(\underline{\bar{r}}\cdot\underline{\bar{r}}\underline{I})\underline{\omega} - \underline{\omega}(\underline{\bar{r}}\otimes\underline{\bar{r}})\underline{\omega} \\
&= \underline{\omega}(\underline{\bar{r}}\cdot\underline{\bar{r}}\underline{I} - \underline{\bar{r}}\otimes\underline{\bar{r}})\underline{\omega}
\end{aligned}$$

$$T = \underbrace{\frac{1}{2}m\underline{\bar{v}}\cdot\underline{\bar{v}}}_{T_{\text{translational}}} + \underbrace{\frac{1}{2}\underline{\omega}\underbrace{\int_{\mathbb{R}}\rho(\underline{\bar{r}}\cdot\underline{\bar{r}}\underline{I} - \underline{\bar{r}}\otimes\underline{\bar{r}})dv}_{\underline{J}^{CM}}\underline{\omega}}_{T_{\text{rotational}}}$$

(8) Prove that parallel axis theorem: $\underline{J}^0 = \underline{J}^{CM} + \underline{J}^{CM/0}$

$$\begin{aligned}
\underline{J}^0 &= \int_{\mathbb{R}} \rho(\underline{r_0}\cdot\underline{r_0}\underline{I} - \underline{r_0}\otimes\underline{r_0})dv \\
&(\underline{r_0} = \underline{x} - \underline{x_0},\ \underline{\bar{r}} = \underline{x} - \underline{\bar{x}},\ \underline{\bar{r}}_0 = \underline{\bar{x}} - \underline{x_0}) \to (\underline{r_0} = \underline{\bar{r}} + \underline{\bar{r_0}}) \\
&= \int_{R} \rho((\underline{\bar{r}} + \underline{\bar{r_0}})\cdot(\underline{\bar{r}} + \underline{\bar{r_0}})\underline{I} - (\underline{\bar{r}} + \underline{\bar{r_0}})\otimes(\underline{\bar{r}} + \underline{\bar{r_0}}))dv \\
&= \int_{\mathbb{R}} \rho(\underline{\bar{r}}\cdot\underline{\bar{r}}\underline{I} - \underline{\bar{r}}\otimes\underline{\bar{r}})dv + \int_{\mathbb{R}} \rho(\underline{\bar{r_0}}\cdot\underline{\bar{r_0}}\underline{I} - \underline{\bar{r_0}}\otimes\underline{\bar{r_0}})dv + 2\underline{\bar{r_0}}\cdot\int_{\mathbb{R}}\rho\underline{\bar{r}}dv\underline{I} \\
&- \underline{\bar{r_0}}\otimes\int_{\mathbb{R}}\rho\underline{\bar{r}}dv - \int_{\mathbb{R}}\rho\underline{\bar{r}}\otimes\underline{\bar{r_0}} \\
&= \underbrace{\int_{\mathbb{R}} \rho(\underline{\bar{r}}\cdot\underline{\bar{r}}\underline{I} - \underline{\bar{r}}\otimes\underline{\bar{r}})dv}_{\underline{J}^{CM}} + \underbrace{\int_{\mathbb{R}} \rho(\underline{\bar{r_0}}\cdot\underline{\bar{r_0}}\underline{I} - \underline{\bar{r_0}}\otimes\underline{\bar{r_0}})dv}_{\underline{J}^{CM/0}=m(\underline{\bar{r_0}}\cdot\underline{\bar{r_0}}\underline{I}-\underline{\bar{r_0}}\otimes\underline{\bar{r_0}})}
\end{aligned}$$

Remark: Note that $\underline{M}^0 = \dot{\underline{\mathcal{H}}}^0$ is reformulated wrt center of mass $\underline{M}^{CM} = \underline{\dot{\mathcal{H}}^{CM}}$ and $\underline{\mathcal{H}^{CM}} = \underline{J}^{CM}\underline{\omega}$. Now the question is whether the following holds: $\underline{\mathcal{H}}^0 = \underline{J}^0\underline{\omega}$ such that we can simplify equations, however this is not true in general. To see this lets see the results from 3:

$$\begin{aligned}
\underline{\mathcal{H}}^0 &= \underline{\mathcal{H}}^{CM} + (\underline{\bar{x}} - \underline{x_0}) \times \underline{P} \\
&= \underline{J}^{CM}\underline{\omega} + \underline{\bar{r_0}} \times m\underline{\bar{v}}
\end{aligned}$$

Now, if $\underline{\bar{v}} = \underline{\omega} \times \underline{\bar{r_0}}$ (not true in general, even in 2D for example only rotating disc without a linear velocity). This holds if we have a disc swinging about the fixed point. Then.

$$\begin{aligned}
\underline{\mathcal{H}}^0 &= \underline{J}^{CM}\underline{\omega} + m\underline{\bar{r_0}} \times (\underline{\omega} \times \underline{\bar{r_0}}) \\
&= \underline{J}^{CM}\underline{\omega} + \underline{J}^{CM/0}\underline{\omega} \\
&= \underline{J}^0\underline{\omega}
\end{aligned}$$

Chapter 5

Linear Elasticity

Here deformations are large(finite) or small(infinitesimal). Linear elasticity assume small deformations. Lets have length of L on a dimension:

$$\underline{x} = \underline{X} + \underline{u} \quad \underline{u}: \text{ displacement vector}$$

$$\underline{F} = \frac{\partial \underline{x}}{\partial \underline{X}} = \underline{I} + \underbrace{\frac{\partial \underline{u}}{\partial \underline{X}}}_{\underline{H}} \quad \underline{H}: \text{ displacement gradient}$$

Linearization

Small deformation means that magnitude of the deformation $|\underline{H}| << 1$ practical way of checking this is $|\underline{u}|/L << 1$, keep in mind that this is true for small rotations and stretches.

Linearized kinematics: retain terms of order $O(\underline{H})$ or $O(|\underline{u}|/L)$, Lets look into right cauchy-green deformation tensor:

$$\underline{F} = \underline{I} + \underline{H}(\text{ already linear})$$

$$\underline{C} = \underline{F}^T\underline{F} \rightarrow \underline{E} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}((\underline{I} + \underline{H}^T)(\underline{I} + \underline{H}) - \underline{I}) = \underbrace{\frac{1}{2}(\underline{H} + \underline{H}^T)}_{\underline{\epsilon}} + \underbrace{\frac{1}{2}\underline{H}^T\underline{H}}_{\text{omit}}$$

$$\underline{\epsilon}: \text{ Infinitesimal strain tensor}$$

For the left cauchy-green tensor, for this we need the inverse of $\underline{F}$ and for that we can use the power series expansion on $(\underline{I} + \underline{H})^{-1}$ and take the linear components:

$$\underline{b} = \underline{FF^T} \rightarrow \underline{e} = \frac{1}{2}(\underline{I} - \underline{b}^{-1}) = \frac{1}{2}(\underline{I} - (\underline{I} - \underline{H}^T)(\underline{I} - \underline{H})) = \underbrace{\frac{1}{2}(\underline{H}^T + \underline{H})}_{\underline{\epsilon}} - \underbrace{\frac{1}{2}\underline{H}^T\underline{H}}_{\text{omit}}$$

Hence when deformations are small, all the strain measures are equal. Then the linearized derivation is:

$$\frac{\partial \underline{u}}{\partial \underline{X}} = \frac{\partial \underline{u}}{\partial \underline{x}}\frac{\partial \underline{x}}{\partial \underline{X}} = \underbrace{\frac{\partial \underline{u}}{\partial \underline{x}}}_{O(\underline{H})} + \underbrace{\frac{\partial \underline{u}}{\partial \underline{x}}\underline{H}}_{O(\underline{H}^2)} \approx \frac{\partial \underline{u}}{\partial \underline{x}}$$

Small strain $\sim \underline{\epsilon} = \underline{H}^{sym}$
Small rotation $\sim ... = \underline{H}^{skw}$

5.1 Linearized Kinetics

$\underline{P} = \hat{\underline{P}}(\underline{F}) \to \underline{0}\hat{\underline{P}}(\underline{I})$, no deformation no stress:

$$J = \det[\underline{F}] = 1 + \text{tr}[H] \to \rho_0 = \rho J \approx \rho$$

$$\underline{T} = \frac{1}{J}\underline{PF^T} \approx \frac{1}{1+\text{tr}[H]}\underline{P}(\underline{I}+\underline{H})^T \approx \underbrace{\underline{P}}_{O(\underline{H})} + \underbrace{\underline{PH^T}}_{O(\underline{H}^2)} \approx \underline{P}$$

$$\underline{S} = \underline{F}^{-1}\underline{P} \approx (\underline{I} - \underline{H})\underline{P} \approx \underline{P}$$

$$\underline{T} \approx \underline{S} \approx \underline{P} = \underline{\sigma} : \text{ infinitesimal stress tensor(symmetric)}$$

All stress measures are equal in the linearized settings.

$$LMB : \text{div}[\underline{\sigma}] + \rho\underline{b} = \rho\ddot{\underline{u}} \leftarrow \underline{\sigma} = \hat{\underline{\sigma}}(\underline{\epsilon})$$

5.2 Material Model(Constitutive Formulation)

Lets use the stretched rod example from before. We had the stretching force $\underline{P}$ and cross section A. In large deformations stress might depend on the history of the deformation and we might not return to original state. In elastic domain we stay inside the linear region and relations are quite simple(Hooke's law):

$$\sigma = E\epsilon$$

where E is the young's modulus. For the 3D we have the following:

$$\underline{\sigma} = \underline{\mathbb{C}\epsilon}$$

where $\underline{\mathbb{C}}$ is a constant fourth order stiffness tensor since it is the most general mapping between two 2nd order tensor.

5.3 Material Symmetry

Currently we have the following:

$$\underbrace{\underbrace{\sigma_{ij}}_{3\times3}}_{6 \text{ indep}} = \underbrace{\underbrace{\mathbb{C}_{ijkl}}_{3\times3\times3\times3}}_{21 \text{ indep}} \underbrace{\underbrace{\epsilon_{kl}}_{3\times3}}_{6 \text{ indep}}$$

Number of independent constants depend on material symmetry. If we were to map these independent variables as matrix and vector products where upper three component is normal components and the bottom three components are shear components:

$$\begin{bmatrix}\sigma_{11}=\sigma_x\\ \sigma_{22}=\sigma_y\\ \sigma_{33}=\sigma_z\\ \tau_{23}=\tau_{yz}\\ \tau_{13}=\tau_{xz}\\ \tau_{12}=\tau_{xy}\end{bmatrix} \rightarrow \begin{bmatrix}\epsilon_{11}=\epsilon_x\\ \epsilon_{22}=\epsilon_y\\ \epsilon_{33}=\epsilon_z\\ 2\epsilon_{23}=\gamma_{yz}\\ 2\epsilon_{13}=\gamma_{xz}\\ 2\epsilon_{12}=\gamma_{xy}\end{bmatrix}$$

Here comes the material symmetry where:

$$\{\epsilon\}_{6\times1} = [S]_{6\times6}\{\sigma\}_{6\times1}$$
$$\{\sigma\}_{6\times1} = [\mathbb{C}]_{6\times6}\{\epsilon\}_{6\times1} \rightarrow [\mathbb{C}] = [S]^{-1}$$

Here S is called the compliance tensor and $\mathbb{C}$ is called the stiffness tensor and both of these are symmetric. Keep in mind that this convention is not same across different fields, this is the voigt notation, another notation is the mandel notation.

5.3.1 Isotropy

Here the compliance matrix has the following properties:

$$[S] = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & \emptyset & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/\mu & 0 & 0 \\ & \emptyset & & 0 & 1/\mu & 0 \\ & & & 0 & 0 & 1/\mu \end{bmatrix}$$

These relations come from the following:

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z))$$
$$\gamma_{xy} = \frac{1}{\mu}\tau_{xy}$$

Here E : Young's Modulus > 0, ν : Poisson Ratio$\in [-1, 1/2]$ for isotropic materials, μ : shear modulus > 0:

$$\mu = \frac{E}{2(1+\nu)}$$

Stiffness matrix is the following:

$$[\mathbb{C}] = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & & & \\ \lambda & 2\mu+\lambda & \lambda & & \emptyset & \\ \lambda & \lambda & 2\mu+\lambda & & & \\ & & & \mu & 0 & 0 \\ & \emptyset & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix}$$

λ : Lame Constant:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Then in the tensor notation:

$$\begin{aligned}
\underline{\sigma} &= \lambda \mathrm{tr}[\underline{\epsilon}]\underline{I} + 2\mu\underline{\epsilon} \\
&= \underbrace{(\lambda + \frac{1}{3})}_{\kappa\text{:bulk modulus} > 0} \quad \mathrm{tr}[\underline{\epsilon}]\underline{I} + 2\mu\underline{\epsilon}^{dev} \\
&= \kappa \mathrm{tr}[\underline{\epsilon}]\underline{I} + 2\mu\underline{\epsilon}^{dev}
\end{aligned}$$

$$\kappa = \frac{E}{3(1-2\nu)}$$

In isotropy, there are only two independent constant. This is the highest level of symmetry. In this sense it is orientation independent. For an isotropic material we can give examples of glass, polymers.

Cubic Symmetry

Single crystal metals have three types of cubic symmetry.

1 If a cube structure is not accompanied by any other atom it is called simple cubic, this is very rare. One example is polonium

2 If the cube structure has a central atom as well it is called body centered cubic structure. Iron and tungsten are such materials.

3 If these atoms are sitting on faces of this cubic structures it is called face centered cubic structure. Aluminum and silver are such materials.

In the cubic symmetry $[S]$ shows the identity of isotropy but $\mu \neq \frac{E}{2(1+\nu)}$ The upper and lower bound for the ν are determined by the μ and κ constants where they are becoming singularities

5.3.2 Orthotropy

There are 9 independent constants. Compliant materials like epoxy need strengthening materials such as glass, carbon, kevlar fibers. This structure in 3D can be supported with different stiffening materials in each axis to have different constants, in the case of fibers are aligning with normal bases(x,y,z axes) we have the off-diagonal elements as 0 but if these fibers did not aligned with these normal vectors we would not get 0 on these entries:

$$[S] = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & & & \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & & \emptyset & \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & & & \\ & & & \frac{1}{\mu_{yz}} & 0 & 0 \\ & \emptyset & & 0 & \frac{1}{\mu_{xz}} & 0 \\ & & & 0 & 0 & \frac{1}{\mu_{xy}} \end{bmatrix}$$

If we were to compare symmetry levels:

isotropy(2) $<$ cubic(3) $<$ transverse isotropy(5) $<$ otrhotropy(9)

Chapter 6

Mechanics of Soft Materials

6.1 Internal Power

LMB in local spatial form dictates:

$$\text{div}[\underline{T}] + \rho\underline{b} = \rho\dot{\underline{v}}$$

Then $\forall \underline{w} = \tilde{\underline{w}}(\underline{x}, t)$:

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \underline{w} \cdot (\rho\dot{\underline{v}} - \text{div}[\underline{T}] - \rho\underline{b})dv \\ &= \int_{\mathbb{R}} \underline{w} \cdot \rho\dot{\underline{v}}dv - \underbrace{\int_{\mathbb{R}} \underline{w} \cdot \text{div}[\underline{T}]dv}_{-\int_{\mathbb{R}} \text{grad}[\underline{w}]\cdot\underline{T}dv + \int_{\partial\mathbb{R}} \underline{w}\cdot\underline{T}\underline{n}da} - \int_{\mathbb{R}} \underline{w} \cdot \rho\underline{b}dv \end{aligned}$$

$$\underbrace{\int_{\mathbb{R}} \underline{w} \cdot \rho\dot{\underline{v}}dv}_{1} + \underbrace{\int_{\mathbb{R}} \text{grad}[\underline{w}]\underline{T}dv}_{2} = \underbrace{\int_{\mathbb{R}} \underline{w} \cdot \rho\underline{b}dv + \int_{\partial\mathbb{R}} \underline{w} \cdot \underline{t}da}_{3}$$

Principle of D'Alambent where $\dot{\underline{v}} = 0$(Quasistatics) links principle of virtual work. Now lets pick $\underline{w} = \underline{v}$:

$$1 = \dot{T} \text{ since } T = \frac{1}{2}\int_{\mathbb{R}} \rho\underline{v} \cdot \underline{v}dv$$
$$3 = \mathcal{P}_{ext}(\text{Power of external forces})$$
$$2 = \mathcal{P}_{int}(\text{Power of internal forces})$$

Hence

$$\mathcal{P}_{ext} = \dot{T} + \mathcal{P}_{int}$$

Hence the $\mathcal{P}_{int}$ is associated with the work needed for deformation.

$$\begin{aligned}
\mathcal{P}_{int} &= \int_{\mathbb{R}} \underline{T} \cdot \underline{L} dv \\
&= \int_{\mathbb{R}_0} \underline{\tau} \cdot \underline{L} dV \\
&= \int_{\mathbb{R}_0} \underline{\tau} \cdot \underline{\dot{F} F^{-1}} dV \\
&= \int_{\mathbb{R}_0} \underline{\tau} \cdot \underline{F^{-T} \dot{F}} dV \\
&= \int_{\mathbb{R}_0} \underbrace{\underline{P} \cdot \underline{\dot{F}}}_{\text{stress power}} dV \\
&= \int_{\mathbb{R}_0} \underline{S} \cdot \underline{F^T \dot{F}} dV \\
&= \int_{\mathbb{R}_0} \underline{S} \cdot (\underline{F^T \dot{F}})^{sym} dV \\
&= \int_{\mathbb{R}_0} \underbrace{\underline{S} \cdot \underline{\dot{E}}}_{\text{stress power}} dV
\end{aligned}$$

6.2 Hyperelasticity

Lets have a space defined by the eigenvalues $\Lambda_i \tilde{\underline{E}}$. Lets have two configuration and the path that links them, from 1 to 2 we have A and from 2 to 1 we have B. We are going to look at the work needed to change to these configurations.

$$\begin{aligned}
work^A_{1-2} &= \int_1^2 \underline{S} \cdot \underline{d\dot{E}} dt \\
&= \int_1^2 \underline{S} \cdot \underline{dE} = \int_1^2 dW \\
work^B_{2-1} &= \int_2^1 \underline{S} \cdot \underline{dE} = \int_2^1 dW \\
&\text{SUM THESE TWO :} \\
0 &= \oint \underline{S} \cdot \underline{dE} = \oint dW
\end{aligned}$$

If we say that this is 0 our material is not dissipating any energy and this equivalent to material being elastic. This integral is path independent. $dW = \underline{S} \cdot \underline{dE} \to \underline{S} = \dfrac{\partial W}{\partial \underline{E}}$. Stress must come from a potential and this potential is W.

The relation $\underline{S} = \dfrac{\partial W}{\partial \underline{E}}$ or $\underline{P} = \dfrac{\partial W}{\partial \underline{F}}$ is called hyperelasticity. $\underline{W} = \underline{\hat{W}}(\underline{E})$ or $\underline{W} = \underline{\hat{W}}(\underline{F})$ and W is strain energy function.

For an isotropic material, the orientation of the three eigenvalues of $\underline{E}$ does not change the stored energy.

$$\underline{E} = \sum_{\alpha} \Lambda_\alpha \underline{V_\alpha} \otimes \underline{V_\alpha}$$
$$\underline{QFQ^T} = \sum_{\alpha} \Lambda_\alpha \underline{QV_\alpha} \otimes \underline{QV_\alpha} \rightarrow \underline{\hat{W}}(\underline{E}) = \underline{\hat{W}}(\underline{QFQ^T})$$

Then an isotropic representation theorem states:

$$\begin{aligned} W &= \bar{W}(\Lambda_1, \Lambda_2, \Lambda_3) \\ &= \hat{W}(I_E, II_E, III_E) \end{aligned}$$

Then we have 3DOF. For an isotropic material stress is found through using cayley hamilton theorem:

$$\begin{aligned} \underline{S} = \frac{\partial W}{\partial \underline{E}} &= \frac{\partial W}{\partial I_E} \underbrace{\frac{\partial I_E}{\partial \underline{E}}}_{\underline{I}} + \frac{\partial W}{\partial II_E} \underbrace{\frac{\partial II_E}{\partial \underline{E}}}_{I_E \underline{I} - \underline{E}} + \frac{\partial W}{\partial III_E} \underbrace{\frac{\partial III_E}{\partial \underline{E}}}_{III_E \underline{E}^{-1}} \\ &= \alpha_0 \underline{I} + \alpha_1 \underline{E} + \alpha_2 \underline{E}^2 \end{aligned}$$

where

$$\alpha_0 = \frac{\partial W}{\partial I_E} + I_E \frac{\partial W}{\partial II_E} + II_E \frac{\partial W}{\partial III_E} \quad \alpha_1 = -\frac{\partial W}{\partial II_E} - I_E \frac{\partial W}{\partial III_E} \quad \alpha_2 = \frac{\partial W}{\partial III_E}$$

In linear elasticity: $\underline{\sigma} = \underline{\mathbb{C}\epsilon}$

$$\mathcal{P}_{ext} = \dot{T} + \mathcal{P}_{int}$$

This mechanical energy balance. We have a stress power:

$$\underline{P} \cdot \underline{\dot{F}} = \underline{S} \cdot \underline{\dot{E}}$$

6.3 Isotropic Strain Energy Functions

6.3.1 Linear Elasticity

In the linear case $\underline{\sigma} = \underline{\mathbb{C}\epsilon}$. Where $\underline{\sigma} = \kappa \mathrm{tr}[\underline{\epsilon}]\underline{I} + 2\mu\underline{\epsilon}^{dev}$. Energy in the small deformation regime is measured, we apply force and record the displacement. This turns out to be

$$\begin{aligned} \omega &= \frac{1}{2}\underline{\sigma} \cdot \underline{\epsilon} = \frac{1}{2}\underline{\epsilon} \cdot \underline{\mathbb{C}\epsilon} \\ &= \frac{1}{2}\kappa \mathrm{tr}[\underline{\epsilon}]\underline{I} \cdot \underline{\epsilon} + \mu\underline{\epsilon}^{dev} \cdot \underline{\epsilon} \\ &= \underbrace{\frac{1}{2}\kappa(\mathrm{tr}[\underline{\epsilon}])^2}_{\omega^{vol}} + \underbrace{\mu\underline{\epsilon}^{dev} \cdot \underline{\epsilon}^{dev}}_{\omega^{shear} = \omega^{dev}} \end{aligned}$$

This is called infinitesimal strain energy function. This is a special case for a general nonlinear formulation. This expression will be a basis for nonlinear formulation. $J = 1 + \text{tr}[\underline{\epsilon}]$. In this expression we have a volumetric-deviatoric decoupling.

(i) Kirchhoff-St. Venant:

$$\underline{S} = \mathbb{C}\underline{E} = \kappa \text{tr}[\underline{\epsilon}]\underline{I} + 2\mu\underline{\epsilon}^{dev}$$

This also holds for anisotropy and we do not need to do additional calculations.

$$\begin{aligned} W &= \frac{1}{2}\underline{S} \cdot \underline{E} \\ &= \frac{1}{2}\underline{E} \cdot \underline{\mathbb{C}E} \\ &= \frac{1}{2}\kappa(\text{tr}[\underline{E}]) + \mu\underline{E^{dev}} \cdot \underline{E^{dev}} \end{aligned}$$

This one is suitable when the strain is small, or stretch is small. In the case of small stretch and large rotations this will still work since $\underline{E}$ filters out the rotations. This is for fabric, thin sheets, ropes.

(ii) Neo-Hooke: This model still incorporates two constant parameters but maintains a highly non linear formulation:

$$W = \underbrace{\frac{1}{2}\kappa(\ln J)^2}_{W^{vol}} + \underbrace{\frac{1}{2}\mu(J^{-2/3}\text{tr}[\underline{C}] - 3)}_{W^{dev}}$$

$$\begin{aligned} \underline{S} &= \frac{\partial W}{\partial \underline{E}} = \frac{\partial W^{vol}}{\partial \underline{E}} + \frac{\partial W^{dev}}{\partial \underline{E}} \\ \frac{\partial W^{vol}}{\partial \underline{E}} &= 2\frac{\partial W^{vol}}{\partial \underline{C}} = 2\kappa \ln J \frac{\partial \ln J}{\partial \underline{C}} = 2\kappa \frac{\ln J}{J}\frac{\partial J}{\partial \underline{C}} = 2\kappa\frac{\ln J}{J}\frac{1}{2}\frac{1}{\sqrt{III_C}}\frac{\partial III_C}{\partial \underline{C}} \\ &= \kappa \ln J\underline{C}^{-1} \\ \frac{\partial W^{dev}}{\partial \underline{E}} &= 2\frac{\partial W^{dev}}{\partial \underline{C}} = \mu(\frac{-2}{3}J^{-5/3}\frac{\partial J}{\partial \underline{C}}\text{tr}[\underline{C}] + J^{-2/3}\frac{\partial \text{tr}[\underline{C}]}{\partial \underline{C}}) \\ &= \mu J^{-2/3}(\frac{-1}{3}\text{tr}[\underline{C}]\underline{C}^{-1} + \underline{I}) \\ \underline{S} &= \kappa \ln J\underline{C}^{-1} + \mu J^{-2/3}(\underline{I} - \frac{1}{3}\text{tr}[\underline{C}]\underline{C}^{-1}) \\ \underline{T} &= \frac{1}{J}\underline{FSF^T} \\ &= \kappa\frac{\ln J}{J}\underline{FC^{-1}F^T} + \mu J^{-5/3}(\underbrace{\underline{FF^T}}_{\underline{b}} - \frac{1}{3}\text{tr}[\underline{b}]\underline{FC^{-1}F^T}) \\ &= \underbrace{\kappa\frac{\ln J}{J}\underline{I}}_{\underline{T}^{sph}} + \underbrace{\mu J^{-5/3}\underline{b}^{dev}}_{\underline{T}^{dev}} \end{aligned}$$

(iii) Ogden:

$$W = W^{vol} + W^{dev}$$

$$W^{vol} = \hat{W}(J) : \begin{cases} \text{from Neo-Hooke, or} \\ \frac{K}{4}(J^2 - 2\ln J - 1) \end{cases}$$

$$W^{dev} = \sum_{p=1}^{M} \frac{\mu_p}{\alpha_p}(\Lambda_1^{\alpha_p} + \Lambda_2^{\alpha_p} + \Lambda_3^{\alpha_p} - 3)$$

Where Λ_is are the principal stretches of $J^{-1/3}\underline{F}$ and $\Lambda_1\Lambda_2\Lambda_3 = 1$. This should return to the linear model so we have a constraint:

$$\sum_{p=1}^{M} \mu_p \alpha_p = 2\mu$$

So we divide the referential representation into M representatino states:

$$\mathbb{R}_0 = \sum_{i=1}^{M} \mathbb{R}_0^{(i)}$$

Chapter 7

Atomic To Continuum Scale Transition

Chapter 8

Navier-Stokes Equations

Any configuration of the fluid can be taken as reference configuration, which implies the memoryless property of the systems. But because of technical and practical reasons we will work in only one configuration.

8.1 Kinematics of Fluid Motion

This is actually the content of fluid dynamics course but we will look into the vorticity term. Recalling RBD:

$$\underline{x} = \underline{QX} + \underline{c} \rightarrow \underline{F} = \underline{Q}$$
$$\underline{L} = \underline{\dot{F}F^T} = \underline{\dot{Q}Q^T} = \underline{\Omega} : \text{ skew tensor and has the axial vector } \underline{\omega}$$
$$\underline{v} = \underline{\bar{v}} + \underline{\bar{r}},\ \underline{\bar{r}} = \underline{v} - \underline{\bar{v}}$$
$$\underline{\bar{r}} = \underline{x} - \underline{\bar{x}} \rightarrow \underline{\dot{\bar{r}}} = \underline{\Omega\bar{r}} = \underline{\omega} \times \underline{r}$$

Consequentially

$$\frac{1}{2}\nabla \times \underline{v} = \underline{\omega} = -\frac{1}{2}e_{ijk}\Omega_{jk}\underline{e_i}$$

In general $\underline{L} = \underline{D} + \underline{W}$ where $\underline{D}$ is symmetric, contributing to stretching and $\underline{W}$ is skew symmetric, contributing to vorticity. Here stretching $\underline{D} = 0$ for RBD since $\underline{L} = \underline{\Omega} = \underline{W}$. Axial vector of $\underline{W}$ is $\underline{w} = -\frac{1}{2}e_{ijk}\underbrace{W_{jk}}_{v_{j,k}-v_{k,j}}\underline{e_i} = \frac{1}{2}\nabla \times \underline{v}$.

Clearly fluid is not a rigid body. Lets choose a spherical volume inside a fluid. Then $\underline{D}$ represents the rate of stretching along the principal directions of $\underline{D}$ and deforms the spheres into ellipsoids. $\underline{W}$ is the rate of rotation about the point of interest, in the context of fluid sphere it is the rotation of the small volume of water.

Now that $\nabla \times \underline{v}$ is vorticity, we can define some states. $\nabla \times \underline{v} = 0$ is irrotational flow, and $\nabla \cdot \underline{v} = 0$ solenoidal flow. If we consider the strain energy function we define more states. If $\underline{v} = \nabla\phi$ then $\nabla \times \underline{v} = 0$ and this is sufficient. It turns out that for irrotational flow $\underline{v} = \nabla\phi$ which is necassary to insure $\nabla \times \underline{v} = \underline{0}$. Then $\phi = \tilde{\phi}(\underline{x}, t)$ is the velocity potential.

8.2 Kinetics of Fluid Motion

From the mass balance we have the following local form $\dot{\rho} + \rho\nabla \cdot \underline{v} = 0$. Here incompressibility dictates that $\dot{\rho} = 0$ which indicates $\nabla \cdot \underline{v} = 0 = \text{tr}[\underline{L}] = \text{tr}[\underline{D}] \rightarrow \underline{D}^{dev} = \underline{D}$ suggesting solenoidal flow.

From the linear momentum balance we have $\nabla \cdot \underline{T} + \rho\underline{b} = \rho\dot{\underline{v}}$. We will look into Linear Elasticity because a newtonian viscous fluids similar to the modeling of stress in linear elasticity. Recall that for isotropic linear elasticity, κ is bulk modulus and μ is shear modulus:

$$\underline{\sigma} = \kappa\text{tr}[\underline{\epsilon}]\underline{I} + w\mu\underline{\epsilon}^{dev}$$
$$\underline{\epsilon} = \underline{\mathcal{H}}^{sym} = \nabla\underline{u}$$

if $\kappa >> \mu \rightarrow$ or $\kappa \rightarrow \infty$ which means that solid objects are incompressible.

$\kappa\text{tr}[\underline{\epsilon}] = -p$: finite $\rightarrow \text{tr}[\underline{\epsilon}] = 0$

$J = 1 + \text{tr}[\underline{\epsilon}] = 1$: no volume change.

So, $\underline{\epsilon}^{dev} = \underline{\epsilon} - \frac{1}{3}\text{tr}[\underline{\epsilon}]\underline{I} \equiv \underline{\epsilon}$:

$$\underline{\sigma} = -p\underline{I} + 2\mu\underline{\epsilon}$$

here $-p$ acts as an lagrange multiplier that enforces a constraint and the constraint here is $\text{tr}[\underline{\epsilon}] = 0$

8.3 Newtonian Viscous Flow

This implies incompressibility, as a consequence volume should not change. In the case of linear elasticity $\underline{\epsilon}$ is the symmetric part of the $(\nabla\underline{u})^{sym}$, in the linear elasticity context $(\nabla\underline{u})^{skew}$ corresponds to rotation and that does not contribute to stress hence the omitting. Going back to the fluid context, lets call viscosity μ:

$$\underline{T} = -p\underline{I} + 2\mu\underline{D},\ \underline{D} = (\nabla\underline{v})^{sym}$$

Here the pressures acts as an lagrange multiplier to enforce the vanishing of $\text{tr}[\underline{D}] = 0$ or $\nabla \cdot \underline{v} = 0$. Here rotations also should not play a role: $(\nabla\underline{v})^{skew} \equiv \underline{W}$ where $\underline{D}^{dev} = \underline{D}$. Here viscosity is a constant μ. In general $\underline{T} = \hat{\underline{T}}(\underline{D})$, nonlinear function of the displacement gradient.

Back to LMB:

$$\nabla \cdot [-p\underline{I} + 2\mu\underline{D}] + \rho\underline{b} = \rho\dot{\underline{v}}$$
$$\begin{aligned}
[-p\underline{I} + 2\mu\underline{D}] &= [-p\delta_{ij} + \mu(v_{i,j} + v_{j,i})]_{,j}\underline{e_i} \\
&= (-p_{,i} + \mu(v_{i,jj} + v_{j,ij}))\underline{e_i} \\
&= (-p_{,i} + \mu(v_{i,jj} + v_{j,ji}))\underline{e_i} \\
&= (-p_{,i} + \mu(\underbrace{\underbrace{v_{i,jj}}_{\nabla\cdot\nabla\underline{v} = \nabla^2\underline{v}}}_{} + \underbrace{(v_{j,j})_{,i}}_{\text{tr}[\underline{L}]=0}))\underline{e_i}
\end{aligned}$$

$$-\nabla p + \mu\nabla^2\underline{v} + \rho\underline{b} = \rho(\frac{\partial\underline{v}}{\partial t} + (\nabla\underline{v})\underline{v})$$

This is called Navier-Stokes equations. Subject to $\nabla \cdot \underline{v} = 0$. Now we have 4 equations and 4 unknowns. 3 equations coming from N-S and one from the constraint. Pressure and the velocity field constitutes 4 unknowns. This equations may seem easy but it is hard to satisfy. There is a numerical challenge in satisfying the constraint. There is a physical challenge in the right hand side of the vectorial equation form which is not linear in $\underline{v}$, this non linearity causes richness in the physics and makes the solution harder.

8.4 Special And Alternative Forms

To get rid off the body force:

$$\begin{aligned} \rho: \text{ constant }, \ \underline{b} = -g\underline{e_3} \rightarrow \rho\underline{b} = \nabla[\rho\underline{b} \cdot \underline{x}] \\ -\nabla \underbrace{(p + \rho g x_3)}_{\tilde{p}\text{: head}} + \mu\nabla^2\underline{v} = \rho\dot{\underline{v}} \end{aligned}$$

From now on we are going to use $p = \tilde{p}$. Now the mass balance dictates $\nabla \cdot \underline{v} = 0$. For the LMB we see that $\underline{T} = \tilde{\underline{T}}(\underline{L}) = \tilde{\underline{T}}(\underline{D})$. Stress is not generated through displacement but velocity.

This is a very specific case where we are considering newtonian viscous flow. Now this leaves us in the equations:

$$\begin{aligned} \nabla \cdot \underline{v} = 0 \\ -\nabla p + \nabla^2\underline{v} = \rho\dot{\underline{v}} \end{aligned}$$

8.4.1 Irrotational (Potential) Flow

$\nabla \times \underline{v} = 0$, $\underline{v} = \nabla\phi$. Mass balance:

$$\nabla \cdot \underline{v} = \nabla^2\phi = 0$$

This may be solvable on its own. Solve for $\tilde{\phi}(\underline{x}, t)$ which also solves for $\underline{v}$ then plug that into LMB and find the pressure $\underline{\tilde{p}}(\underline{x}, t)$. However this example is not a realistic assumption because none of the fluids are irrotational. But this solution may give an idea about the characteristics about the flow field. We can use this solution and add some correction terms to it to get the nearly good solution for other systems.

8.4.2 Invicid (Euler) Flow

$-\nabla p = \rho\dot{\underline{v}}$ which makes $\mu = 0$. This works in the flow where the reynolds number is high, this can be due to high velocity. Meaningful approximation where the pressure variations are mostly die to inertial effects. In other words away from the boundary layers. E.g. flow around an airplane.

8.4.3 Creeping (Stokes) Flow

In this flow reynolds number is quite low. $\underline{v}$ is small, now

$$\dot{\underline{v}} \approx \frac{\partial \underline{v}}{\partial t} = \underline{0}$$

Negligible inertial effects:

$$-\nabla p + \nabla^2 \underline{v} = \underline{0}$$

Which makes the problem linear. Examples are flow in porous media, microfluidic devices.

8.4.4 Dimensionless Form

Now let $t^* = \frac{t}{t_0}$, $\underline{x}^* = \frac{\underline{x}}{L_0}$, $\underline{v}^* = \frac{\underline{v}}{v_0}$, $p^* = \frac{p}{p_0}$.

$$\nabla \cdot \underline{v} = \frac{\partial v_i}{\partial x_i} = 0$$

$$-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2} = \rho \left(\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j \right)$$

We are going to choose $p_0 = \rho v_0^2$, $t_0 = \frac{L_0}{v_0}$. L_0 corresponds to reference length chose according to your structure, and v_0 is order of magnitude that velocity experiences in terms of change.

$$\frac{\partial v_i^*}{\partial x_i^*} = 0$$

$$-\frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re} \frac{\partial^2 v_i^*}{\partial x_j^{*2}} = \frac{\partial v_i^*}{\partial t^*} + \frac{\partial v_i^*}{\partial x_j^*} v_j^*$$

$$Re = \frac{\rho v_0 L_0}{\mu}$$

8.4.5 Turbulence

Now the reynolds numbers characterizes the nature of the flow, there are some critical reynolds numbers $Re > Re_{critical}(O(10^3 - 10^4)) >> 1$. What we observe in turbulence is that we are looking at the structures in the flow fields, it is always 3D

1. 3D

2. Observing a lot of randomness. Random: irregular, chaotic behaviour, you replicate the experiment side to side you get different results, unpredictable

3. Nonlinear: convective inertia dominates, cannot simplify this. This embodies the major difficulty with the turbulence.

4. Diffusive: momentum and temperature mixing occurs rapidly. In turbulent fields fluid particles advects these attributes to the further volumes making it easier to diffuse over the medium.

5. Vorticity: flow field has identifiable rotational flow structures, eddies. These occur at many different length and time scales. Length and time scale are inversely associated in these structures.

6. Dissipative: this is related to velocity gradient, there is always dissipation in fluids but in turbulent ones viscosity dissipates energy at small length scales which leads to large gradients. To maintain a turbulent flow we must put energy into it.

Very strong fluctuations of $\underline{v}$ in t occurs. We are going to average it out:

$$a(t) = \underbrace{\bar{a}}_{\text{mean}} + \underbrace{\tilde{a}(t)}_{\text{fluctuations}}$$

$$\bar{a} = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} a(t)dt = < a >$$

For the velocity we are going to decompose it in the same way, $\underline{v} = \underline{\bar{v}} + \underline{\tilde{v}}(t)$. We can make the average as a function of time and make it as a window average function that will smooth out the velocity and filter out the high frequency fluctuations. We can also have the number of different experiments and take the average of them which is the ensemble averaging operation.

Eventually $< \tilde{a} >= 0$ and this implies $< \underline{\tilde{v}} >= \underline{0}$. Lets remind N-S equations:

$$-\nabla p + \mu\nabla^2\underline{v} = \rho\underline{\dot{v}}$$
$$\nabla\cdot\underline{v} = 0$$

Now we will average out these equations as well. Velocity fields is highly oscillatory in time and space. If we want to solve this with grids or particles we should have tiny spaces. In the smooth versions we dont have these. We can resolve thse equations to mean version, we are going the discuss the possibility of the following altered equations:

$$-\nabla\bar{p} + \mu\nabla^2\underline{\bar{v}} = \rho\dot{\underline{\bar{v}}}$$
$$\nabla\cdot\underline{\bar{v}} = 0$$

Starting with the mass balance:

$$< \nabla\cdot\underline{v} > = \nabla\cdot < \underline{v} >$$
$$= \nabla\cdot\underline{\bar{v}} = 0$$
$$\nabla\cdot\underline{\tilde{v}} = 0$$

For the linear momentum balance we have the following:

$$< -\nabla p > + < \mu\nabla^2\underline{v} > = < \rho\underline{\dot{v}} >$$
$$-\nabla < p > + \mu\nabla^2 < \underline{v} > = \rho\frac{\partial < \underline{v} >}{\partial t} + \rho < \nabla\underline{v}\underline{v} >$$
$$-\nabla < p > + \mu\nabla^2 < \underline{v} > = \rho\frac{\partial < \underline{v} >}{\partial t} + \rho < \nabla(\underline{\bar{v}} + \underline{\tilde{v}})(\underline{\bar{v}} + \underline{\tilde{v}}) >$$
$$-\nabla < p > + \mu\nabla^2 < \underline{v} > = \rho\frac{\partial < \underline{\bar{v}} >}{\partial t} + \rho[< \nabla\underline{\bar{v}}\underline{\bar{v}} > + < \nabla\underline{\tilde{v}}\underline{\bar{v}} > + < \nabla\underline{\bar{v}}\underline{\tilde{v}} > + < \nabla\underline{\tilde{v}}\underline{\tilde{v}} >]$$
$$-\nabla\bar{p} + \mu\nabla^2\underline{\bar{v}} = \rho\frac{\partial\underline{\bar{v}}}{\partial t} + \rho[\nabla\underline{\bar{v}}\underline{\bar{v}} + < \nabla\underline{\tilde{v}}\underline{\tilde{v}} >]$$

using the divergence constraint we can see that $<\nabla \underline{\tilde{v}}\underline{\tilde{v}}> \equiv \nabla \cdot [<\underline{\tilde{v}} \otimes \underline{\tilde{v}}>]$. Then we ended up with the averaged MB:

$$\nabla \cdot \underline{v} = 0$$

And averaged LMB:

$$-\nabla \bar{p} + \mu \nabla^2 \underline{\bar{v}} = \underbrace{\rho \frac{\partial \underline{\bar{v}}}{\partial t} + \rho \nabla \underline{\bar{v}}\underline{\bar{v}}}_{\rho \dot{\underline{\bar{v}}}} + \rho \nabla \cdot [<\underline{\tilde{v}} \otimes \underline{\tilde{v}}>]$$

LMB can be expressed alternatively as:

$$\nabla \cdot \underline{\bar{T}} = \rho \dot{\underline{\bar{v}}}$$
$$\underline{\bar{T}} = -\bar{p}\underline{I} + 2\mu \underbrace{\underline{\bar{D}}}_{(\nabla \underline{\bar{v}})^{sym}} \underbrace{-\rho <\underline{\tilde{v}} \otimes \underline{\tilde{v}}>}_{\underline{R}:\text{additional}}$$

Where $\underline{R}$ is Reynolds stress tensor. These averaging called Reynolds Averaging and these equations are called Reynolds Averaged Navier-Stokes(RANS). If the additional Reynolds stress tensor is small it can be neglected. To solve RANS we must model $\underline{R}$, these can be modeled in terms of mean fields $\underline{R} = \underline{R}(\bar{v})$. This modeling becomes easier if we invoke the isotropy, then it must follow the form of $\underline{R} = r\underline{I}$. In this case we observe similar rotational structures that is isotropic turbulence. If not applicable we have anisotropic turbulence. In this case we must model it so that it is manageable, which is cheaper solution, coarse resolution in time and space. It turns out that RANS is the simplest model. There is RANS in one extreme and full solution of N-S in other extreme. This solution is the DNS which direct numerical simulations. Observing turbulence can be achieved with this expensive method. Turbulence is the graveyard of theories, they dont always work but there is not a universal simplification to DNS.